

WEYL QUANTIZATION OF DEGREE 2 SYMPLECTIC GRADED MANIFOLDS

MELCHIOR GRÜTZMANN, JEAN-PHILIPPE MICHEL, AND PING XU

ABSTRACT. Let S be a spinor bundle of a pseudo-Euclidean vector bundle (E, g) of even rank. We introduce a new filtration on the algebra $\mathcal{D}(M, S)$ of differential operators on S . As main property, the associated graded algebra $\text{gr } \mathcal{D}(M, S)$ is isomorphic to the algebra $\mathcal{O}(\mathcal{M})$ of functions on \mathcal{M} , where \mathcal{M} is the symplectic graded manifold of degree 2 canonically associated to (E, g) . Accordingly, we define the Weyl quantization on \mathcal{M} as a map $\mathcal{WQ}_\hbar : \mathcal{O}(\mathcal{M}) \rightarrow \mathcal{D}(M, S)$, and prove that \mathcal{WQ}_\hbar satisfies all desired usual properties. As an application, we obtain a bijection between Courant algebroid structures $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$, that are encoded by Hamiltonian generating functions on \mathcal{M} , and skew-symmetric Dirac generating operators $D \in \mathcal{D}(M, S)$. The operator D^2 gives a new invariant of $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$, which generalizes the square norm of the Cartan 3-form of a quadratic Lie algebra. We study in detail the particular case of E being the double of a Lie bialgebroid (A, A^*) .

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1. INTRODUCTION

This paper is devoted to the study of Weyl quantization of degree 2 symplectic graded manifolds, and its application to Courant algebroids.

In searching for the Lie algebroid analogue of Drinfeld's double of Lie bialgebras [10], Liu–Weinstein–Xu introduced the notion of Courant algebroids [21]. The Courant algebroid axioms were later reformulated by Roytenberg [29] in terms of Dorfman brackets. However these axioms still remain mysterious, and various attempts have been made in order to understand Courant algebroids in a more conceptual and transparent way. One approach was through a degree 3 Hamiltonian function in a degree 2 symplectic graded manifold. When a Courant algebroid E is the double $A \oplus A^*$ of a Lie bialgebroid (A, A^*) , Roytenberg [29] proved that the Courant algebroid structure on E is indeed equivalent to a degree 3 Hamiltonian generating function Θ on the symplectic graded manifold $T^*[2](A[1])$ satisfying the equation $\{\Theta, \Theta\} = 0$. Following an idea of Weinstein, Roytenberg [30] and Ševera [32] extended this result to arbitrary Courant algebroids and proved that there is essentially a bijection between Courant algebroids and degree 3 Hamiltonian generating functions on degree 2 symplectic graded manifolds. By a graded manifold we always mean a \mathbb{N} -graded manifold.

Around the same time, in 2001, in an unpublished manuscript [1], Alekseev–Xu took a different approach in terms of Dirac generating operators, an analogue of Kostant's cubic Dirac operators [20]. Alekseev–Xu's approach was motivated by the following basic example [6] due to Cabras–Vinogradov. Let M be a manifold, and $\mathfrak{X}(M) \cong \Gamma(TM)$ be the Lie algebra of vector fields on M . The idea is to extend the Lie bracket on $\mathfrak{X}(M)$ to sections of the bundle $E = TM \oplus T^*M$. Observe that sections of E act on the space of differential forms $\Omega(M)$ by contraction and by exterior multiplication, respectively,

$$(X + \alpha) \cdot \mu := \iota_X \mu + \alpha \wedge \mu,$$

where $\alpha \in \Gamma(T^*M) = \Omega^1(M)$, $X \in \mathfrak{X}(M)$ and $\mu \in \Omega(M)$. This action turns $\Omega(M)$ into a Clifford module of the Clifford bundle $\text{Cl}(E)$, where E is equipped with the standard bilinear form:

$$(X_1 + \alpha_1, X_2 + \alpha_2) = \frac{1}{2}(\langle \alpha_1, X_2 \rangle + \langle \alpha_2, X_1 \rangle),$$

and the Clifford generating relation is $xy + yx = 2(x, y)$.

Using the de Rham differential $d : \Omega^\bullet(M) \rightarrow \Omega^{\bullet+1}(M)$, one can form the *derived bracket* [17] on sections of E :

$$(1.1) \quad \llbracket e_1, e_2 \rrbracket := [[d, e_1], e_2], \quad \forall e_1, e_2 \in \Gamma(E),$$

where both sides are viewed as operators on $\Omega(M)$. It is straightforward to check that Eq. (1.1) coincides with the Dorfman bracket of the standard Courant algebroid $TM \oplus T^*M$.

Observe that $\Omega(M)$ is indeed a real spinor bundle of $(TM \oplus T^*M, \langle \cdot, \cdot \rangle)$. For a general Courant algebroid $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$, Alekseev–Xu [1] proved that there exist cubic Dirac type generating operators, acting on a certain spinor bundle of (E, g) , which play exactly the same role as the de Rham differential operator d does in Cabras–Vinogradov’s approach to the standard Courant algebroid¹. Thus a natural question arises: is there any relation between Hamiltonian generating functions and Dirac generating operators of a Courant algebroid?

To answer this question, we are naturally led to the study of Weyl quantization on symplectic graded manifolds of degree 2. The classical Weyl quantization formula is the prototypical example of a quantization map: to each polynomial function on the classical phase space $T^*\mathbb{R}^n$, it assigns a differential operator on \mathbb{R}^n . More precisely, the Weyl quantization maps polynomials in the coordinates (x^i, p_i) to differential operators on x^i , and is defined as the symmetrization map such that $p_i \mapsto \frac{\hbar}{i} \frac{\partial}{\partial x^i}$ and $x^i \mapsto m_{x^i}$ (multiplication by x^i). Weyl quantization has been generalized to various contexts and plays a role in different branches of mathematics, e.g. harmonic analysis [12], pseudo-differential symbolic calculus [16], formal [2] and strict [27] deformation quantizations. More specifically, we are concerned by the extension of Weyl quantization to smooth manifolds and supermanifolds. Using an affine connection on M , Underhill has built a Weyl quantization on the symplectic manifold T^*M [35]. Considering in addition a connection on a vector bundle $V \rightarrow M$, Widom [37] has obtained more generally a quantization map of the form $\mathcal{Q}_\hbar^M : \mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{End } V) \rightarrow \mathcal{D}(M, V)$, where $\mathcal{D}(M, V)$ is the algebra of differential operators on V , see also [4]. In the case where $V = S$ is the spinor bundle of a Riemannian spin manifold (M, g) , Getzler [13] has used such a map \mathcal{Q}_\hbar^M to get a Weyl quantization on the symplectic supermanifold $T^*M \oplus \Pi TM$ and then prove the index theorem for the Riemannian Dirac operator. See [36] for the case where $S = \Omega(M)$. Other related quantization schemes have been applied to even symplectic supermanifolds, see e.g. [5, 25, 24].

In our situation, as a first step, by taking a metric preserving connection, we can write any degree 2 symplectic graded manifold as $T^*[2]M \oplus E[1]$, where E is a vector bundle

¹ To the best of our knowledge, Cabras–Vinogradov’s contributions to Courant brackets seem to have been overlooked by the community so far.

over M equipped with a pseudo-metric g . According to [28], the degree 2 symplectic form on $T^*[2]M \oplus E[1]$ can be written explicitly in terms of the canonical symplectic structure on $T^*[2]M$, the pseudo-metric g and the chosen connection. Our first main result is to construct Weyl type quantization for this graded symplectic manifold $T^*[2]M \oplus E[1]$ by a combination of Clifford quantization and classical Weyl quantization. This generalizes the previous works [13, 36]. More precisely, we assume that (E, g) is of even rank and admits a spinor bundle S , i.e., $\text{End } S \cong \text{Cl}(E)$ with $\text{Cl}(E) = \text{Cl}(E) \otimes \mathbb{C}$ the complex Clifford bundle. Then, we introduce an increasing filtration of the algebra $\mathcal{D}(M, S)$: $\mathcal{D}_0(M, S) \subset \mathcal{D}_1(M, S) \subset \dots \subset \mathcal{D}_k(M, S) \subset \dots$ satisfying the conditions

$$\begin{aligned} D_k(M, S) \cdot \mathcal{D}_l(M, S) &\subseteq \mathcal{D}_{k+l}(M, S), \\ [\mathcal{D}_k(M, S), \mathcal{D}_l(M, S)] &\subseteq \mathcal{D}_{k+l-2}(M, S), \end{aligned} \quad \forall k, l \in \mathbb{N}.$$

As a consequence, the associated graded algebra $\text{gr } \mathcal{D}(M, S) = \bigoplus_{k \in \mathbb{N}} \mathcal{D}_k(M, S) / \mathcal{D}_{k-1}(M, S)$, is a graded commutative Poisson algebra of degree -2 , which we prove to be isomorphic to $\mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1])$. Such an isomorphism enables us to define the principal symbol $\sigma_k : \mathcal{D}_k(M, S) \rightarrow \mathcal{O}_k^{\mathbb{C}}(T^*[2]M \oplus E[1])$, $\forall k \in \mathbb{N}$, exactly in the same way as in the classical case. We prove that the Weyl quantization \mathcal{WQ}_h establishes an isomorphism $\mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1]) \rightarrow \mathcal{D}(M, S)$, which is the right inverse of the principal symbol map, and satisfies all the usual desired properties. Note that the filtration we introduce on $\mathcal{D}(M, S)$ is distinct from both the usual filtration by the order of derivations and the Getzler's filtration (see [13]). In a certain sense, this is the only filtration which turns $\text{gr } \mathcal{D}(M, S)$ into a graded commutative Poisson algebra, with non-degenerate Poisson bracket. Roughly speaking, it assigns degree 2 to derivations and degree 1 to sections in $\Gamma(E) \subset \Gamma(\text{Cl}(E))$.

The second part of the paper is devoted to the application of Weyl quantization to Courant algebroids. The consideration of a specific spinor bundle \mathbb{S} , obtained by twisting S by a certain line bundle, allows us to define a conjugation map and an adjoint operation on $\mathcal{D}(M, \mathbb{S})$. A Dirac generating operator is then defined as a real operator in $\mathcal{D}_3(M, \mathbb{S})$, which is odd and squares to a function on the base manifold. Following an idea of Ševera [32], we prove that, for a given Courant algebroid, there exists a unique skew-symmetric Dirac generating operator, and moreover the Weyl quantization map \mathcal{WQ} establishes a bijection between Hamiltonian generating functions and skew-symmetric Dirac generating operators. This is our second main result. As an application, by considering the square of the unique skew-symmetric Dirac generating operator, we obtain a new Courant algebroid invariant. We prove that this new invariant, as a function on the base manifold, is a natural extension of the square norm of the Cartan 3-form of a quadratic Lie algebra. As another consequence, we recover a result of Chen–Stiénon [8] regarding Dirac generating operators for Lie bialgebroids, which gives an equivalent description of Lie bialgebroid compatibility condition. In this case, $E = A \oplus A^*$ and there are two natural twisted spinor bundles, each of them admit a Dirac generating operator. When the Lie bialgebroid corresponds to a generalized complex structure, they coincide with the ∂ and $\bar{\partial}$ -operators of the generalized complex structure [7].

Some remarks are in order. We learned recently that Li-Bland and Meinrenken also obtained similar results in their study of Dirac generating operators [23].

Notations. Finally, we list the notations used throughout the paper.

$\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the set of non-negative integers, $\mathbb{N}^\times = \{1, 2, \dots\}$ the set of positive ones and $i = \sqrt{-1}$. Tensor products over the algebra of real numbers are denoted by \otimes or $\otimes_{\mathbb{R}}$, whereas tensor products over the algebra $C^\infty(M)$ are denoted by \otimes_{C^∞} . For a vector space (or a vector bundle) V , symmetric and skew-symmetric tensor products are denoted by $\mathcal{S}V$ and $\wedge V$ respectively. We use the Einstein's summation convention without further comments.

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2. SYMPLECTIC GRADED MANIFOLDS OF DEGREE 2

We recall in this section some standard materials concerning symplectic graded manifolds. Our presentation is based mainly on [30, 31], enriched with the work of Rothstein on symplectic supermanifolds [28].

2.1. Definition. A *graded manifold* \mathcal{M} is a smooth manifold M endowed with a sheaf of \mathbb{N} -graded algebras \mathcal{O} such that, for every contractible open set $U \subset M$, the algebra of functions $\mathcal{O}(U)$ is isomorphic to $C^\infty(U) \otimes \mathcal{S}V$, for a fixed \mathbb{N}^\times -graded vector space $V = \bigoplus_{i \in \mathbb{N}^\times} V_i$. Here, $\mathcal{S}V = \bigotimes_{i \in 2\mathbb{N}^\times} \mathcal{S}V_i \otimes \wedge V_{i-1}$ is the graded symmetric tensor algebra of V and the grading of $\mathcal{O}(U)$ is induced by the one of V . In particular, the degree 0 component of $\mathcal{O}(U)$ is $C^\infty(U)$. The algebra of global sections of \mathcal{O} is called the algebra of functions on \mathcal{M} and denoted by $\mathcal{O}(\mathcal{M})$. Coordinates on U together with a graded basis of V form a local coordinate system (x^i) on \mathcal{M} . The grading on the algebra of functions, $\mathcal{O}(\mathcal{M}) = \bigoplus_{k \in \mathbb{N}} \mathcal{O}_k(\mathcal{M})$, is then given by the decomposition of $\mathcal{O}(\mathcal{M})$ into eigenspaces of the Euler vector field on \mathcal{M} :

$$\epsilon = w(x^i) x^i \frac{\partial}{\partial x^i},$$

where $w(x^i) \in \mathbb{N}$ denotes the degree of the coordinate x^i , i.e., $w(x^i) = 0$ if x^i is a coordinate on U and $w(x^i) = j$ if $x^i \in V_j$.

Given a smooth vector bundle $E \rightarrow M$ and a positive integer $k \in \mathbb{N}^\times$, by $E[k]$, we denote the graded manifold with base M , whose algebra of functions is

$$\mathcal{O}(E[k]) = \begin{cases} \Gamma(\wedge E^*) & \text{for } k \text{ odd,} \\ \Gamma(\mathcal{S}E^*) & \text{for } k \text{ even.} \end{cases}$$

Here sections of E^* are assigned the degree k . The graded manifold $E[k]$ is also called a shifted vector bundle. We will mostly consider graded manifolds build out of shifted vector bundles.

A *symplectic graded manifold* of degree n is a graded manifold \mathcal{M} endowed with a symplectic form of degree n , i.e. a closed non-degenerate 2-form ω , whose Lie derivative along the Euler vector field ϵ satisfies $L_\epsilon \omega = n\omega$. The algebra of functions $\mathcal{O}(\mathcal{M})$ admits a Poisson bracket of degree $-n$, i.e., $\{\mathcal{O}_k(\mathcal{M}), \mathcal{O}_l(\mathcal{M})\} \subset \mathcal{O}_{k+l-n}(\mathcal{M})$ for all $k, l \in \mathbb{N}$. The Poisson bracket is graded skew-symmetric and satisfies the graded Jacobi identity, namely

$$\begin{aligned} \{F, G\} &= -(-1)^{(k-n)(l-n)} \{G, F\}, \\ \{F, \{G, H\}\} &= \{\{F, G\}, H\} + (-1)^{(k-n)(l-n)} \{G, \{F, H\}\}, \end{aligned}$$

for all $F \in \mathcal{O}_k(\mathcal{M})$, $G \in \mathcal{O}_l(\mathcal{M})$ and $H \in \mathcal{O}(\mathcal{M})$.

2.2. Example. In [28], Rothstein gives a description of symplectic supermanifolds in terms of the following data: a pseudo-Euclidean vector bundle (E, g) over an ordinary symplectic manifold and a metric connection ∇ on (E, g) , i.e., a connection on E satisfying $\nabla g = 0$. In the sequel, we will adapt his construction to the graded context and obtain a symplectic structure of degree 2 on the Whitney sum $T^*[2]M \oplus E[1]$ over M .

Proposition 2.1 ([28]). *Let $E \rightarrow M$ be a smooth vector bundle, endowed with a pseudo-Euclidean metric g and a metric connection ∇ . Then, the graded manifold $T^*[2]M \oplus E[1]$ admits an exact symplectic 2-form of degree 2:*

$$(2.1) \quad \omega_{g, \nabla} := d\alpha \quad \text{where} \quad \alpha = \pi_1^* \alpha_0 + \pi_2^* \beta.$$

Here π_1, π_2 are the canonical projections on $T^*[2]M$ and $E[1]$ respectively, α_0 is the Liouville 1-form on $T^*[2]M$ and $\beta \in \Omega^1(E[1])$ is the 1-form on $E[1]$ which annihilates on the horizontal subspace of $TE[1]$, corresponding to the connection ∇ , and satisfies $\beta_e(v_e) = \frac{1}{2}g_{\pi(e)}(v_e, e)$ for all vertical tangent vector fields $v_e \in T_e E[1]$.

In what follows, we make repeated use of the identification $E \cong E^*$, induced by the metric g , and of the identifications below, without mentioning them explicitly:

$$\begin{aligned} \mathcal{O}_0(T^*[2]M \oplus E[1]) &\cong C^\infty(M), \\ \mathcal{O}_1(T^*[2]M \oplus E[1]) &\cong \Gamma(E), \\ \mathcal{O}_2(T^*[2]M \oplus E[1]) &\cong \Gamma(TM \oplus \wedge^2 E). \end{aligned}$$

The spaces $C^\infty(M)$, $\Gamma(E)$ and $\mathfrak{X}(M) \cong \mathfrak{X}(M)$ generate the entire algebra of functions on $T^*[2]M \oplus E[1]$. Hence, by the Leibniz rule, the symplectic structure on $T^*[2]M \oplus E[1]$

can be characterized in terms of the following Poisson brackets:

$$(2.2) \quad \begin{aligned} \{X, f\} &= X(f), & \{h, f\} &= 0, \\ \{X, \xi\} &= \nabla_X \xi, & \{f, \xi\} &= 0, \\ \{X, Y\} &= [X, Y] + R(X, Y), & \{\xi, \eta\} &= g(\xi, \eta), \end{aligned}$$

where $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Gamma(E)$ and $f, h \in C^\infty(M)$. Since ∇ is a metric connection, its curvature $R(X, Y)$ defines an element in $\Gamma(\wedge^2 E)$, i.e., a degree 2 function on $T^*[2]M \oplus E[1]$. From (2.2) it is simple to see that the bracket $\{\cdot, \cdot\}$ is indeed of degree -2 .

Remark 2.2. According to Eq. (2.2), $E[1]$ is a Poisson submanifold of $T^*[2]M \oplus E[1]$, whose Poisson bracket is induced by the pseudo-metric g .

For a vector bundle $E \rightarrow M$, the cotangent bundle $T^*[2]E[1]$ is naturally a degree 2 symplectic graded manifold. Let $\pi_E : T^*[2]E[1] \rightarrow E[1]$ and $\pi_{E^*} : T^*[2]E[1] \cong T^*[2]E^*[1] \rightarrow E^*[1]$ be the natural projections. They combine into a map:

$$(2.3) \quad \tilde{\pi} : T^*[2]E[1] \longrightarrow (E \oplus E^*)[1].$$

A connection ∇ on E gives rise to a horizontal distribution on the tangent bundle TE , which in turn induces a surjective submersion

$$(2.4) \quad \pi_\nabla : T^*[2]E[1] \longrightarrow T^*[2]M.$$

Putting together the maps $\tilde{\pi}$ and π_∇ , we obtain a map:

$$\tilde{\Xi}_\nabla : T^*[2]E[1] \longrightarrow T^*[2]M \oplus (E \oplus E^*)[1].$$

It is simple to check that $\tilde{\Xi}_\nabla$ is a diffeomorphism. Note that $E \oplus E^*$ is a pseudo-Euclidean bundle over M with the duality pairing. The connection ∇ on E induces a connection on $E \oplus E^*$, which is compatible with the duality pairing. According to Proposition 2.1, these data induce a degree 2 symplectic structure on $T^*[2]M \oplus (E \oplus E^*)[1]$.

Lemma 2.3. *The map $\tilde{\Xi}_\nabla$ is a symplectic diffeomorphism.*

Proof. Since $\tilde{\Xi}_\nabla$ is a diffeomorphism, it suffices to prove that it is a Poisson map. As $\mathcal{O}(T^*[2]M \oplus (E \oplus E^*)[1])$ is generated by $C^\infty(M)$, $\Gamma(E \oplus E^*)$ and $\mathfrak{X}(M)$, it is thus sufficient to check that $\tilde{\Xi}_\nabla^*$ preserves the Poisson brackets on these spaces.

It is simple to see that

$$(2.5) \quad \begin{aligned} (\tilde{\Xi}_\nabla)^* f &= (\pi_E)^* f, & (\tilde{\Xi}_\nabla)^* \eta &= (\pi_E)^* \eta, \\ (\tilde{\Xi}_\nabla)^* X &= (\pi_\nabla)^* X, & (\tilde{\Xi}_\nabla)^* \xi &= (\pi_{E^*})^* \xi, \end{aligned}$$

where $f \in C^\infty(M)$, $\eta \in \Gamma(E)$, $\xi \in \Gamma(E^*)$ and $X \in \mathfrak{X}(M)$. Here, both $(\pi_E)^* f$ and $(\pi_E)^* \eta$ are fiberwise constant functions on $T^*[2]E[1]$ and both $(\pi_\nabla)^* X$ and $(\pi_{E^*})^* \xi$ are fiberwise linear functions on $T^*[2]E[1]$, corresponding to the vector fields ∇_X and $\iota_\xi = g(\xi, \cdot)$ on $E[1]$. Hence, the Poisson brackets in $T^*[2]E[1]$ of the four functions in Eq. (2.5) are easily computed. In $T^*[2]M \oplus (E \oplus E^*)[1]$, the Poisson brackets of f , η , ξ and X are given by Eq. (2.2), with E ,

g and ∇ replaced by $E \oplus E^*$, the duality pairing and the induced connection. The conclusion follows from a straightforward verification. \square

2.3. Classification of symplectic graded manifolds of degree 2. According to [30], any pseudo-Euclidean vector bundle (E, g) determines a degree 2 symplectic graded manifold \mathcal{M} , which is defined to be the fiber product $(T^*[2]E[1]) \times_{(E \oplus E^*)[1]} E[1]$. Thus, we have the following commutative diagram:

$$(2.6) \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{i_{\mathcal{M}}} & T^*[2]E[1] \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ E[1] & \xrightarrow{i} & (E \oplus E^*)[1] \end{array}$$

Here $\tilde{\pi}$ is defined as in (2.3), and i is the diagonal-like map $\psi \mapsto \psi \oplus \frac{1}{2}g(\psi, \cdot)$. Since i is an isometric embedding, $i_{\mathcal{M}}$ must be an embedding as well. It is simple to see that the restriction to \mathcal{M} of the canonical symplectic form on $T^*[2]E[1]$ is non-degenerate. Therefore \mathcal{M} is a symplectic submanifold of $T^*[2]E[1]$. It is known that, up to an isomorphism, every degree 2 symplectic graded manifold indeed arises in this way. We refer the interested reader to [30] for details.

Let ∇ be a connection on E . Then composing $i_{\mathcal{M}}$ with the map π_{∇} , as defined in (2.4), we obtain a map

$$\pi_{\nabla} \circ i_{\mathcal{M}} : \mathcal{M} \longrightarrow T^*[2]M.$$

Together with the natural projection $\pi : \mathcal{M} \rightarrow E[1]$, we obtain a map

$$\Xi_{\nabla} : \mathcal{M} \longrightarrow T^*[2]M \oplus E[1].$$

It is simple to check that Ξ_{∇} is a diffeomorphism. The following result is known to experts and sketched in [30].

Theorem 2.4. *Let ∇ be a metric connection on (E, g) . Then the map Ξ_{∇} is a symplectic diffeomorphism, where $T^*[2]M \oplus E[1]$ is endowed with the symplectic structure (2.1).*

Proof. Consider the map

$$i_T = \text{id}_{T^*[2]M} \oplus i : T^*[2]M \oplus E[1] \longrightarrow T^*[2]M \oplus (E \oplus E^*)[1].$$

It is simple to check that i_T is a symplectic embedding. By definition, we have

$$(2.7) \quad \tilde{\Xi}_{\nabla} = \pi_{\nabla} \oplus \tilde{\pi} \quad \text{and} \quad \Xi_{\nabla} = \pi_{\nabla} \circ i_{\mathcal{M}} \oplus \pi.$$

According to the relation $\tilde{\pi} \circ i_{\mathcal{M}} = i \circ \pi$ (see diagram (2.6)), we have the following commutative diagram:

$$\begin{array}{ccc} T^*[2]E[1] & \xrightarrow{\tilde{\Xi}_{\nabla}} & T^*[2]M \oplus (E \oplus E^*)[1] \\ i_{\mathcal{M}} \uparrow & & \uparrow i_T \\ \mathcal{M} & \xrightarrow{\Xi_{\nabla}} & T^*[2]M \oplus E[1] \end{array}$$

Since both $i_{\mathcal{M}}$ and i_T are symplectic embeddings and $\tilde{\Xi}_{\nabla}$ is a symplectic diffeomorphism, it follows that Ξ_{∇} must be a symplectic diffeomorphism. \square

As a consequence, the symplectic graded manifolds $(T^*[2]M \oplus E[1], \omega_{g, \nabla})$, associated to different metric connections, are all isomorphic. They provide minimal symplectic realization of the Poisson manifold $E[1]$ (see [30]).

3. THE ALGEBRA $\mathcal{D}(M, S)$ OF SPINOR DIFFERENTIAL OPERATORS

After recalling some basic materials regarding Clifford algebras and spinor bundles (see e.g. [3, 34, 22]), we introduce a new filtration on the algebra $\mathcal{D}(M, S)$ of spinor differential operators and determine its associated graded algebra. By considering a specific spinor bundle \mathbb{S} , we obtain in addition two involutions on $\mathcal{D}(M, \mathbb{S})$.

3.1. Clifford and spinor bundles. Let (E, g) be a pseudo-Euclidean vector bundle of even rank over a smooth manifold M . The real *Clifford bundle* $\text{Cl}(E)$ is a bundle of associative algebras, whose fiber at $x \in M$ is isomorphic to the real Clifford algebra

$$\text{Cl}(E_x) := \left(\bigotimes E_x / \mathcal{I} \right),$$

where \mathcal{I} is the ideal generated by $\xi_1(x) \otimes \xi_2(x) + \xi_2(x) \otimes \xi_1(x) - 2g(\xi_1(x), \xi_2(x))$, $\forall \xi_1, \xi_2 \in \Gamma(E)$. In the sequel, we mainly use the complex Clifford bundle $\mathbb{Cl}(E) := \text{Cl}(E) \otimes \mathbb{C}$ and refer to it simply as the Clifford bundle.

The Clifford bundle inherits a natural filtration, $\mathbb{Cl}(E) = \bigcup_{k \in \mathbb{N}} \mathbb{Cl}_k(E)$, and a natural \mathbb{Z}_2 -grading,

$$\mathbb{Cl}(E) = \mathbb{Cl}^+(E) \oplus \mathbb{Cl}^-(E),$$

where $\Gamma(\mathbb{Cl}_k(E))$ is spanned by products of at most k sections of E and $\mathbb{Cl}^+(E)$ (resp. $\mathbb{Cl}^-(E)$) is spanned by products of even (resp. odd) number of sections of E . Through the metric g , the algebra $\Gamma(\wedge E \otimes \mathbb{C})$ can be identified with $\mathcal{O}^{\mathbb{C}}(E[1])$, the algebra of complex valued functions on $E[1]$. Moreover, g induces a Poisson bracket $\{\cdot, \cdot\}$ on $E[1]$ (see Remark 2.2). There is a standard \mathbb{C} -linear isomorphism, called the Clifford quantization map:

$$(3.1) \quad \gamma : \mathcal{O}^{\mathbb{C}}(E[1]) \longrightarrow \Gamma(\mathbb{Cl}(E)).$$

It extends the canonical embedding $\Gamma(E) \hookrightarrow \Gamma(\mathbb{Cl}(E))$ by skew-symmetrization and satisfies

$$(3.2) \quad \gamma(\{\mu, \cdot\}) = \frac{1}{2}[\gamma(\mu), \gamma(\cdot)], \quad \forall \mu \in \Gamma(E \oplus \wedge^2 E).$$

We refer the interested reader to [3, 24, 22] for more details.

By a *spinor bundle*, we mean a complex vector bundle S such that $\text{End } S \cong \mathbb{Cl}(E)$.

Example 3.1. Let A be a vector bundle, and $E = A \oplus A^*$. Let g be the following pairing on E :

$$g(\zeta_1 + \eta_1, \zeta_2 + \eta_2) = \frac{1}{2}\langle \zeta_1, \eta_2 \rangle + \frac{1}{2}\langle \zeta_2, \eta_1 \rangle, \quad \forall \zeta_1, \zeta_2 \in \Gamma(A), \eta_1, \eta_2 \in \Gamma(A^*),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. Then $\wedge A^* \otimes \mathbb{C}$ (or $\wedge A \otimes \mathbb{C}$) is a spinor bundle of (E, g) , and the Clifford action is given by

$$(3.3) \quad \gamma(\zeta)\phi = \iota_\zeta \phi \quad \text{and} \quad \gamma(\eta)\phi = \eta \wedge \phi, \quad \forall \zeta \in \Gamma(A), \eta \in \Gamma(A^*), \phi \in \Gamma(\wedge A^* \otimes \mathbb{C}).$$

Note that $\wedge A^*$ is a real spinor bundle, i.e., $\text{End } \wedge A^* \cong \text{Cl}(E)$.

Remark 3.2. There are certain topological obstructions to the existence of a spinor bundle, which depend on the signature of the metric g . The existence of a real spinor bundle is a further topological constraint on the vector bundle $(E, g) \rightarrow M$.

In what follows, we will always assume that a spinor bundle exists, and use the algebra isomorphisms below without mentioning them explicitly

$$\Gamma(\wedge E \otimes \mathbb{C}) \cong \mathcal{O}^{\mathbb{C}}(E[1]) \quad \text{and} \quad \Gamma(\text{End } S) \cong \Gamma(\text{Cl}(E)).$$

Let ∇^E be a connection on E . It induces a connection on the exterior bundle $\wedge E$ which is compatible with the exterior product, i.e., $\nabla_X^E(\xi_1 \wedge \xi_2) = (\nabla_X^E \xi_1) \wedge \xi_2 + \xi_1 \wedge (\nabla_X^E \xi_2)$, $\forall X \in \mathfrak{X}(M), \xi_1, \xi_2 \in \Gamma(\wedge E)$. If ∇^E is a metric connection, it also induces a connection on the Clifford bundle $\text{Cl}(E)$ which is compatible with the Clifford multiplication. The quantization map (3.1) intertwines the induced connections on $\wedge E \otimes \mathbb{C}$ and $\text{Cl}(E)$. That is,

$$\gamma(\nabla_X^E \eta) = \nabla_X^E \gamma(\eta), \quad \forall \eta \in \mathcal{O}^{\mathbb{C}}(E[1]), X \in \mathfrak{X}(M).$$

Definition 3.3. Assume that (E, g) admits a spinor bundle S . A connection ∇^S on S is called a spinor connection if there exists a metric connection ∇^E on E satisfying the following compatibility condition:

$$(3.4) \quad [\nabla_X^S, \gamma(\eta)] = \gamma(\nabla_X^E \eta),$$

for all $X \in \mathfrak{X}(M)$ and $\eta \in \mathcal{O}^{\mathbb{C}}(E[1])$. In this case, (∇^E, ∇^S) is called a compatible pair of connections.

For compatible connections (∇^E, ∇^S) , the induced connections on $\text{Cl}(E)$ and $\text{End } S$ are intertwined by the isomorphism $\text{Cl}(E) \cong \text{End } S$.

Lemma 3.4. Assume that (E, g) admits a spinor bundle S . Then,

- there always exist compatible connections (∇^E, ∇^S) ;
- there exists $r \in \Omega^2(M) \otimes \mathbb{C}$ such that the curvatures of ∇^E and ∇^S satisfy

$$(3.5) \quad R^S = \frac{1}{2} \gamma(R^E) + r \text{id}_S;$$

- any two pairs of compatible connections $(\tilde{\nabla}^E, \tilde{\nabla}^S)$ and (∇^E, ∇^S) are related as follows

$$\tilde{\nabla}^E - \nabla^E = \{\varpi, \cdot\} \quad \text{and} \quad \tilde{\nabla}^S - \nabla^S = \frac{1}{2} \gamma(\varpi) + \nu \text{id}_S,$$

where $\varpi \in \Omega^1(M, \wedge^2 E)$ and $\nu \in \Omega^1(M) \otimes \mathbb{C}$. Here, $\{\cdot, \cdot\}$ stands for the Poisson bracket on $E[1]$ and we have $\{\varpi(X), \xi\} = -\iota_\xi \varpi(X)$, $\forall X \in \mathfrak{X}(M), \xi \in \Gamma(E)$.

Proof. The existence of a metric connection ∇^E is classical. Such a connection induces a connection on $\mathbb{C}l(E) \cong \text{End } S$. According to [34], the latter is always induced by a connection ∇^S on S . Thus (∇^E, ∇^S) is a pair of compatible connections.

Let $X, Y \in \mathfrak{X}(M)$. Since ∇^E preserves the metric, its curvature can be identified with $R^E \in \Omega^2(M, \wedge^2 E)$, via the following relation

$$\left(\nabla_X^E \nabla_Y^E - \nabla_Y^E \nabla_X^E - \nabla_{[X, Y]}^E \right) \xi = \{R^E(X, Y), \xi\}, \quad \forall \xi \in \Gamma(E).$$

The same relation holds after replacing $\xi \in \Gamma(E)$ by $\eta \in \Gamma(\wedge E \otimes \mathbb{C})$. Using the compatibility condition (3.4), we deduce that

$$[R^S(X, Y), \gamma(\eta)] - \gamma(\{R^E(X, Y), \eta\}) = 0, \quad \forall \eta \in \Gamma(\wedge E \otimes \mathbb{C}).$$

Hence, by Eq. (3.2), we have

$$[R^S(X, Y) - \frac{1}{2}\gamma(R^E(X, Y)), \gamma(\eta)] = 0, \quad \forall \eta \in \Gamma(\wedge E \otimes \mathbb{C}).$$

The center of $\Gamma(\mathbb{C}l(E))$ being the algebra $C^\infty(M) \otimes \mathbb{C}$, Eq. (3.5) follows.

We now compare two pairs of compatible connections $(\tilde{\nabla}^E, \tilde{\nabla}^S)$ and (∇^E, ∇^S) . First, we have $\tilde{\nabla}^E - \nabla^E = \mu$, for some $\mu \in \Omega^1(M, \text{End } E)$. As both connections preserve the metric g , so does μ and we get $\mu = \{\varpi, \cdot\}$ with $\varpi \in \Omega^1(M, \wedge^2 E)$. Similarly, we have $\tilde{\nabla}^S - \nabla^S = A$ with $A \in \Omega^1(M, \text{End } S)$. The compatibility condition (3.4) implies that $[A, \cdot] = \frac{1}{2}[\gamma(\varpi), \cdot]$ on $\Gamma(\text{End } S)$. This yields $A = \frac{1}{2}\gamma(\varpi) + \nu$ for some $\nu \in \Omega^1(M) \otimes \mathbb{C}$. \square

3.2. The filtered algebra $\mathcal{D}(M, S)$ and its associated graded algebra. Assume that (E, g) admits a spinor bundle S . The algebra $\mathcal{D}(M, S)$, of differential operators on S , is a subalgebra of $\text{End}(\Gamma(S))$ generated by $\Gamma(\text{End } S) \cong \Gamma(\mathbb{C}l(E))$ and the covariant derivatives ∇_X , where ∇ is any connection on S and X ranges over all vector fields on M . There is a \mathbb{Z}_2 -grading on $\mathcal{D}(M, S)$, inherited from $\mathbb{C}l(E)$,

$$(3.6) \quad \mathcal{D}(M, S) = \mathcal{D}^+(M, S) \oplus \mathcal{D}^-(M, S),$$

where $\mathcal{D}^\pm(M, S)$ is generated by the covariant derivatives ∇_X and $\Gamma(\mathbb{C}l^\pm(E))$. By $[\cdot, \cdot]$, we denote the graded commutator:

$$[A, B] = A \circ B - (-1)^{|A||B|} B \circ A,$$

for any $A, B \in \mathcal{D}(M, S)$ of the parity $|A|, |B| \in \{0, 1\}$, respectively.

We now introduce a sequence of subspaces of $\mathcal{D}(M, S)$ inductively as follows. Set $\mathcal{D}_{-1}(M, S) := \{0\}$, and

$$(3.7) \quad \mathcal{D}_k(M, S) := \{D \in \mathcal{D}(M, S) \mid \forall \xi \in \Gamma(E), [D, \gamma(\xi)] \in \mathcal{D}_{k-1}(M, S)\}.$$

Note that the ordinary filtration on $\mathcal{D}(M, S)$, given by the differential order, can be defined in a similar fashion replacing $\Gamma(E)$ by $C^\infty(M)$.

Proposition 3.5. *Let ∇^S be a spinor connection on S . For all $k \in \mathbb{N}$, we have*

$$(3.8) \quad \mathcal{D}_k(M, S) = \text{span}\{\gamma(\eta) \circ \nabla_{X_1}^S \cdots \nabla_{X_m}^S \mid 2m + \deg(\eta) \leq k\},$$

where $X_1, \dots, X_m \in \mathfrak{X}(M)$ and $\deg(\eta)$ denotes the degree of $\eta \in \mathcal{O}^{\mathbb{C}}(E[1])$.

Proof. The proof is by induction. Set

$$\widetilde{\mathcal{D}}_k(M, S) = \text{span}\{\gamma(\eta) \circ \nabla_{X_1}^S \cdots \nabla_{X_m}^S \mid 2m + \deg(\eta) \leq k\}.$$

By definition, $\mathcal{D}_{-1}(M, S) = \widetilde{\mathcal{D}}_{-1}(M, S) = \{0\}$. Assume that $\mathcal{D}_k(M, S) = \widetilde{\mathcal{D}}_k(M, S)$ holds for a given $k \geq -1$. Since ∇^S is a spinor connection, there exists a metric connection ∇^E on E such that the following identities hold:

$$(3.9) \quad [\nabla_X^S, \gamma(\xi)] = \gamma(\nabla_X^E \xi) \quad \text{and} \quad [\gamma(\eta), \gamma(\xi)] = 2g(\eta, \xi), \quad \forall \xi, \eta \in \Gamma(E), X \in \mathfrak{X}(M).$$

Hence, if $D \in \widetilde{\mathcal{D}}_{k+1}(M, S)$, we have $[D, \gamma(\xi)] \in \widetilde{\mathcal{D}}_k(M, S)$ for all $\xi \in \Gamma(E)$. The inclusion $\widetilde{\mathcal{D}}_{k+1}(M, S) \subset \mathcal{D}_{k+1}(M, S)$ follows. To prove the converse inclusion we need a lemma.

Lemma 3.6. *Let (x^i) be a local coordinate system on M . If $\alpha = (\alpha_1, \dots, \alpha_k)$ is a multi-index, ∇_α^S stands for the composition $\nabla_{\alpha_1}^S \circ \cdots \circ \nabla_{\alpha_k}^S$, where $\nabla_i^S := \nabla_{\frac{\partial}{\partial x^i}}^S$. Any operator $D \in \mathcal{D}(M, S)$ locally admits a unique linear decomposition*

$$D = \sum_{\kappa, \alpha} \gamma(\eta_\kappa^\alpha) \nabla_\alpha^S,$$

where $\eta_\kappa^\alpha \in \Gamma(\wedge^\kappa E)$. In the sum above, κ runs over \mathbb{N} and α runs over ordered multi-indices of arbitrary length $|\alpha| = k \in \mathbb{N}$, i.e., $\alpha = (\alpha_1, \dots, \alpha_k)$ and $1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq \dim M$.

Proof. It is well-known that $\mathcal{D}(M, S)$ is a locally free $C^\infty(M)$ -module, generated as an algebra by (∇_i^S) and $\gamma(\eta)$, with $\eta \in \Gamma(\wedge^\kappa E)$, $\kappa \in \mathbb{N}$. The result follows then from Eqns (3.4)–(3.5). \square

We suppose now that $D \in \mathcal{D}_{k+1}(M, S)$. There exists a minimum $l \in \mathbb{N}$ such that $D \in \widetilde{\mathcal{D}}_l(M, S)$. We prove that $l - 1 \leq k$. By Lemma 3.6, we have $D = \sum_{\kappa+2|\alpha| \leq l} \gamma(\eta_\kappa^\alpha) \nabla_\alpha^S$. From the definition of $\mathcal{D}_{k+1}(M, S)$, we deduce that $[D, \gamma(\xi)] \in \widetilde{\mathcal{D}}_k(M, S)$ and $\gamma(\xi) \cdot [D, f] = [D, \gamma(\xi)f] - [D, \gamma(\xi)]f \in \widetilde{\mathcal{D}}_k(M, S)$. These relations read respectively as

$$(3.10) \quad \sum_{\kappa+2|\alpha| \leq l} [\gamma(\eta_\kappa^\alpha), \gamma(\xi)] \cdot \nabla_\alpha^S + \gamma(\eta_\kappa^\alpha) \cdot [\nabla_\alpha^S, \gamma(\xi)] \in \widetilde{\mathcal{D}}_k(M, S), \quad \forall \xi \in \Gamma(E),$$

$$(3.11) \quad \sum_{\kappa+2|\alpha| \leq l} \gamma(\xi) \gamma(\eta_\kappa^\alpha) \cdot [\nabla_\alpha^S, f] \in \widetilde{\mathcal{D}}_k(M, S), \quad \forall \xi \in \Gamma(E), f \in C^\infty(M).$$

Consider κ and α such that η_κ^α is non-vanishing, $\kappa + 2|\alpha| = l$ and α is maximal. The term $[\gamma(\eta_\kappa^\alpha), \gamma(\xi)] \cdot \nabla_\alpha^S$ is then clearly independent of the other terms in Eq. (3.10), hence it pertains to $\widetilde{\mathcal{D}}_k(M, S)$. If $\kappa \neq 0$, there exists $\xi \in \Gamma(E)$ such that $0 \neq [\gamma(\eta_\kappa^\alpha), \gamma(\xi)] \in \Gamma(\wedge^{\kappa-1} E)$, and we deduce that $l - 1 \leq k$. If $\kappa = 0$, the same conclusion follows from Eq. (3.11). \square

By Proposition 3.5, we have $\gamma(\xi) \in \mathcal{D}_1(M, S)$ and $\nabla_X^S \in \mathcal{D}_2(M, S)$, if $\xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. Moreover, the operators in $\mathcal{D}_0(M, S)$ are multiplication by functions in $C^\infty(M) \otimes \mathbb{C}$. The following proposition is a direct consequence of Eqns (3.7)–(3.8).

Proposition 3.7. *The subspaces $(\mathcal{D}_k(M, S))_{k \in \mathbb{N}}$ define an increasing filtration of the algebra $\mathcal{D}(M, S)$, namely*

$$\begin{aligned} \mathcal{D}_0(M, S) &\subset \mathcal{D}_1(M, S) \subset \cdots \subset \mathcal{D}_k(M, S) \subset \cdots, \\ \mathcal{D}(M, S) &= \bigcup_{k \in \mathbb{N}} \mathcal{D}_k(M, S), \\ \mathcal{D}_k(M, S) \cdot \mathcal{D}_l(M, S) &\subseteq \mathcal{D}_{k+l}(M, S), \quad \forall k, l \in \mathbb{N}. \end{aligned}$$

Moreover, the filtration satisfies the additional property

$$[\mathcal{D}_k(M, S), \mathcal{D}_l(M, S)] \subseteq \mathcal{D}_{k+l-2}(M, S), \quad \forall k, l \in \mathbb{N}.$$

Remark 3.8. Note that in [25, 24], the same choice of filtration on $\mathcal{D}(M, S)$ was made, whereas in [13] and [36], a different choice was used: both Clifford generators $\gamma(\xi)$, $\xi \in \Gamma(E)$, and covariant derivatives ∇_X^S , $X \in \mathfrak{X}(M)$, are of order 1. In the more standard filtration, however, the Clifford generators normally have order 0.

As a consequence of Proposition 3.7, the graded algebra

$$\text{gr } \mathcal{D}(M, S) = \bigoplus_{k \in \mathbb{N}} \mathcal{D}_k(M, S) / \mathcal{D}_{k-1}(M, S),$$

is a graded commutative Poisson algebra of degree -2 . Moreover, it is isomorphic to $\mathcal{D}(M, S)$ as $C^\infty(M)$ -module. By ς , we denote the canonical projection $\varsigma : \mathcal{D}(M, S) \rightarrow \text{gr } \mathcal{D}(M, S)$.

Theorem 3.9. *Let S be a spinor bundle of (E, g) and (∇^E, ∇^S) a pair of compatible connections. There exists a unique isomorphism of graded commutative Poisson algebras:*

$$\mathfrak{S}_\nabla : \text{gr } \mathcal{D}(M, S) \longrightarrow \mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1]),$$

satisfying

$$(3.12) \quad (\mathfrak{S}_\nabla \circ \varsigma)(f) = f, \quad (\mathfrak{S}_\nabla \circ \varsigma)\left(\frac{1}{\sqrt{2}}\gamma(\xi)\right) = \xi, \quad (\mathfrak{S}_\nabla \circ \varsigma)(\nabla_X^S) = X,$$

for all $f \in C^\infty(M)$, $\xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. Moreover, the isomorphism \mathfrak{S}_∇ only depends on the connection ∇^E .

Proof. The algebras $\mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1])$ and $\text{gr } \mathcal{D}(M, S)$ are generated by $\Gamma(E)$ and $\mathfrak{X}(M)$ over $C^\infty(M)$ (see Proposition 3.5). Hence, an algebra morphism \mathfrak{S}_∇ satisfying Eq. (3.12) is an isomorphism and it must be unique if it exists. As a result, it suffices to prove the existence of \mathfrak{S}_∇ locally. We use canonical coordinates (x^i, p_i) on T^*M , and write $\nabla_i^S := \nabla_{\frac{\partial}{\partial x^i}}^S$. We locally define a map \mathfrak{S}_∇ by setting

$$\mathfrak{S}_\nabla \left(\gamma(\eta) \nabla_\alpha^S \right) = (\sqrt{2})^{\deg(\eta)} \eta p_\alpha,$$

where $\eta \in \Gamma(\wedge E)$, $\alpha = (\alpha_1, \dots, \alpha_k)$ is an ordered multi-index and $p_\alpha = p_{\alpha_1} \cdots p_{\alpha_k}$. According to Lemma 3.6, the map \mathfrak{S}_∇ is well-defined. Moreover, by Eq. (3.9), this is indeed an algebra morphism which satisfies Eq. (3.12).

To prove that \mathfrak{S}_∇ is a Poisson map, it suffices to check that \mathfrak{S}_∇ preserves the Poisson bracket on generators, i.e., on elements in $C^\infty(M) \oplus \Gamma(E) \oplus \mathfrak{X}(M)$. For any $f, g \in C^\infty(M)$,

$\xi, \eta \in \Gamma(E)$, and $X, Y \in \mathfrak{X}(M)$, we have the following graded commutators of differential operators

$$(3.13) \quad \begin{aligned} [\nabla_X^S, f] &= X(f), & [g, f] &= 0, \\ [\nabla_X^S, \gamma(\xi)] &= \gamma(\nabla_X^E \xi), & [g, \gamma(\xi)] &= 0, \\ [\nabla_X^S, \nabla_Y^S] &= \nabla_{[X, Y]}^S + R^S(X, Y), & [\gamma(\xi), \gamma(\eta)] &= 2g(\xi, \eta), \end{aligned}$$

where we have used Eq. (3.4). Writing the curvature R^S as in Eq. (3.5) and applying the map $\mathfrak{S}_\nabla \circ \varsigma$, it is immediate to see that these commutators coincide with the generating relations of the Poisson bracket as given in Eq. (2.2). Therefore, \mathfrak{S}_∇ is indeed a Poisson map.

According to Lemma 3.4, two spinor connections, compatible with the same metric connection ∇^E , satisfy $\tilde{\nabla}^S - \nabla^S = \nu \in \Omega^1(M) \otimes \mathbb{C}$. However, different ν do not modify the defining relations for \mathfrak{S}_∇ in Eq. (3.12). This concludes the proof of the theorem. \square

Definition 3.10. We define the principal symbol map $\sigma_k : \mathcal{D}_k(M, S) \rightarrow \mathcal{O}_k^{\mathbb{C}}(T^*[2]M \oplus E[1])$, $\forall k \in \mathbb{N}$, as the composition $\mathfrak{S}_\nabla \circ \varsigma$.

Note that the principal symbol map σ_k depends on the choice of the connection ∇^E .

Proposition 3.11. Let $k, l \in \mathbb{N}$. The principal symbol maps satisfy the following properties:

$$(3.14) \quad \begin{aligned} \sigma_{k+l}(AB) &= \sigma_k(A)\sigma_l(B), \\ \sigma_{k+l-2}([A, B]) &= \{\sigma_k(A), \sigma_l(B)\}, \end{aligned}$$

for all $A \in \mathcal{D}_k(M, S)$ and $B \in \mathcal{D}_l(M, S)$.

Proof. This is a direct consequence of Theorem 3.9. \square

Remark 3.12. It is helpful to compare our filtration with those in the texts [13, 36], as shown in the table below:

Filtration	order $\gamma(\xi)$	order ∇_X^S	$\text{gr } \mathcal{D}(M, S)$
standard filtration	0	2	$\mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{End } S)$
filtration in [13, 36]	2	2	$\left(\Gamma(\otimes TM) \otimes_{C^\infty} \mathcal{O}^{\mathbb{C}}(E[1]) \right) / \mathcal{J}$
filtration in (3.7)	1	2	$\mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1])$

where $\xi \in \Gamma(E)$, $X, Y \in \mathfrak{X}(M)$ and $\mathcal{J} = (X \otimes Y - Y \otimes X - R^E(X, Y))$. As before, we have $R^E(X, Y) \in \Gamma(\wedge^2 E)$. Note that our filtration is the only one for which $\text{gr } \mathcal{D}(M, S)$ is a graded commutative algebra.

Recall that, according to Theorem 2.4, there exists a symplectic diffeomorphism $\Xi_\nabla : \mathcal{M} \rightarrow T^*[2]M \oplus E[1]$. It is natural to expect the following

Proposition 3.13. The map $((\Xi_\nabla)^* \circ \mathfrak{S}_\nabla) : \text{gr } \mathcal{D}(M, S) \rightarrow \mathcal{O}^{\mathbb{C}}(\mathcal{M})$ is an isomorphism of graded Poisson algebras, which is independent of the metric connection ∇^E used in defining Ξ_∇ and \mathfrak{S}_∇ .

Proof. Let (∇^E, ∇^S) be a pair of compatible connections, and $\Xi_\nabla, \mathfrak{S}_\nabla$ their associated maps as defined in Theorems 2.4 and 3.9. Let $\rho_\nabla = (\Xi_\nabla)^* \circ \mathfrak{S}_\nabla$. Since both $(\Xi_\nabla)^*$ and \mathfrak{S}_∇ are isomorphisms of graded Poisson algebras, so is their composition ρ_∇ . It remains to prove that ρ_∇ is independent of the choice of the connections (∇^E, ∇^S) . Since ρ_∇ is an algebra isomorphism, it suffices to prove the assertion on generators of $\text{gr } \mathcal{D}(M, S)$, which are of the following three types: f , $\varsigma \circ \gamma(\xi)$ and $\varsigma(\nabla_X^S)$, $\forall f \in C^\infty(M)$, $\xi \in \Gamma(E)$, $X \in \mathfrak{X}(M)$. By definition, both maps $(\Xi_\nabla)^*$ and \mathfrak{S}_∇ are independent of the choice of connections when applying to elements f and $\varsigma \circ \gamma(\xi)$. Thus it remains to show that ρ_∇ is also independent of the choice of connections on the elements $\varsigma(\nabla_X^S)$, $\forall X \in \mathfrak{X}(M)$.

Assume that $(\tilde{\nabla}^E, \tilde{\nabla}^S)$ is another pair of compatible connections. From Eqns (2.7) and (3.12), it follows that

$$(3.15) \quad \begin{aligned} \rho_\nabla \circ \varsigma(\nabla_X^S) &= (\pi_\nabla \circ i_\mathcal{M})^* l_X, \\ \rho_{\tilde{\nabla}} \circ \varsigma(\tilde{\nabla}_X^S) &= (\pi_{\tilde{\nabla}} \circ i_\mathcal{M})^* l_X, \end{aligned}$$

where l_X denotes the fiber linear function on $T^*[2]M$ corresponding to $X \in \mathfrak{X}(M)$. According to Lemma 3.4, there exists $\varpi \in \Omega^1(M, \wedge^2 E)$ and $\nu \in \Omega^1(M) \otimes \mathbb{C}$ such that

$$(3.16) \quad \tilde{\nabla}_X^S - \nabla_X^S = \frac{1}{2}\gamma(\varpi(X)) + \nu(X),$$

$$(3.17) \quad \tilde{\nabla}_X^E - \nabla_X^E = \{\varpi(X), \cdot\}.$$

Eq. (3.17) implies that

$$(3.18) \quad (\pi_{\tilde{\nabla}} \circ i_\mathcal{M})^* l_X - (\pi_\nabla \circ i_\mathcal{M})^* l_X = \pi^* \varpi(X).$$

Therefore, by Eqns (3.15) and (3.18), we have

$$\rho_{\tilde{\nabla}} \circ \varsigma(\tilde{\nabla}_X^S) = \rho_\nabla \circ \varsigma(\nabla_X^S) + \pi^* \varpi(X).$$

On the other hand, Eq. (3.16) implies that

$$\rho_\nabla \circ \varsigma(\tilde{\nabla}_X^S) = \rho_\nabla \circ \varsigma(\nabla_X^S) + \pi^* \varpi(X).$$

As a consequence, we conclude that the equality $\rho_{\tilde{\nabla}} = \rho_\nabla$ holds on any element $\varsigma(\tilde{\nabla}_X^S)$, $\forall X \in \mathfrak{X}(M)$. This concludes the proof of the proposition. \square

3.3. Two involutions on $\mathcal{D}(M, \mathbb{S})$. Assume the line bundle $(\det S^*)^{1/N}$ exists, with N being the rank of S . Consider the *twisted spinor bundle* $\mathbb{S} := S \otimes (\det S^*)^{1/N} \otimes |\wedge^{\text{top}} T^* M|^{1/2}$, where $|\wedge^{\text{top}} T^* M|^{1/2}$ is the trivializable line bundle of half-densities on M . A connection ∇ on TM induces a connection on all density bundles. Together with a spinor connection ∇^S on S , it yields an *induced spinor connection* on the twisted spinor bundle \mathbb{S} , denoted by $\nabla^{\mathbb{S}}$. Since \mathbb{S} is a spinor bundle, the algebra $\mathcal{D}(M, \mathbb{S})$, of differential operators acting on \mathbb{S} , satisfies in particular all the results proved in Section 3.2 for the algebra of spinor differential operators. Considering \mathbb{S} rather than an arbitrary spinor bundle allows for further structure on $\mathcal{D}(M, \mathbb{S})$.

First, we build an adjoint operation on the algebra $\mathcal{D}(M, \mathbb{S})$. Let $U \subset M$ be a contractible open subset. A pseudo-Hermitian pairing on $\mathbb{S}|_U$ is a fiberwise non-degenerate sesquilinear

map

$$(3.19) \quad \langle \cdot, \cdot \rangle_U : \Gamma(\mathbb{S}|_U) \times \Gamma(\mathbb{S}|_U) \longrightarrow \Gamma(|\wedge^{\text{top}} T^*U|) \otimes \mathbb{C},$$

which satisfies $\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle$ for all $\phi, \psi \in \Gamma(\mathbb{S}|_U)$.

Proposition 3.14. *Up to multiplication by a scalar $a \in \mathbb{R}^\times$, there exists a unique pseudo-Hermitian pairing $\langle \cdot, \cdot \rangle_U$ on $\mathbb{S}|_U$ satisfying the following properties:*

- (i) $\nabla_X (\langle \phi, \psi \rangle_U) = \langle \nabla_X^\mathbb{S} \phi, \psi \rangle_U + \langle \phi, \nabla_X^\mathbb{S} \psi \rangle_U$,
- (ii) $\langle \gamma(\xi)\phi, \psi \rangle_U = \langle \phi, \gamma(\xi)\psi \rangle_U$,

for arbitrary $X \in \Gamma(TU)$, $\xi \in \Gamma(E|_U)$ and $\phi, \psi \in \Gamma(\mathbb{S}|_U)$.

Moreover, $\langle \cdot, \cdot \rangle_U$ is independent of the choices of connections ∇ and $\nabla^\mathbb{S}$ involved in (i). It is called a spinor pairing.

Proof. Choosing an orthonormal basis (ξ^i) of sections of (E, g) over U , we obtain a trivialization of the bundle $E|_U \cong U \times \mathbb{R}^n$, on which g becomes a constant metric. This in turn induces a trivialization of $\text{Cl}(E|_U) \cong \text{End } \mathbb{S}|_U$. By choosing a nowhere vanishing section $\phi \in \Gamma(\mathbb{S}|_U)$, then we obtain a trivialization $\mathbb{S}|_U \cong U \times \mathbb{R}^N$, under which the operators $\gamma(\xi^i) \in \Gamma(\text{End}(\mathbb{S}|_U)) \cong C^\infty(U, \text{End } \mathbb{R}^N)$ are constant endomorphisms, for all $i = 1, \dots, n$. Consider the induced trivialization of the line bundle $(\det S^*|_U)^{1/N}$ and pick up a trivialization of $|\wedge^{\text{top}} T^*U|$. Then, we can identify the space of sections of $|\wedge^{\text{top}} T^*U|$ with $C^\infty(U)$, and the space of sections of \mathbb{S} with $C^\infty(U, \mathbb{R}^N)$. Under such identifications, the connections ∇ and $\nabla^\mathbb{S}$ can be written as follows:

$$(3.20) \quad \begin{aligned} \nabla_X v &= Xv + \nu_0(X)v, & \forall v \in \Gamma(|\wedge^{\text{top}} T^*U|) &\cong C^\infty(U), \\ \nabla_X^\mathbb{S} \phi &= X\phi + \frac{1}{2}\gamma(\varpi(X))\phi + \frac{1}{2}\nu_0(X)\phi, & \forall \phi \in \Gamma(\mathbb{S}|_U) &\cong C^\infty(U, \mathbb{R}^N), \end{aligned}$$

where $\nu_0 \in \Omega^1(U)$ and $\varpi \in \Omega^1(U, \wedge^2 E|_U)$ (see Lemma 3.4). If Property (ii) holds, it is then simple to check that Property (i) is equivalent to

$$(3.21) \quad X(\langle \phi, \psi \rangle_U) = \langle X\phi, \psi \rangle_U + \langle \phi, X\psi \rangle_U, \quad \forall \phi, \psi \in C^\infty(U, \mathbb{R}^N), X \in \Gamma(TU).$$

The latter equation means that $\langle \cdot, \cdot \rangle_U$ is a constant pairing in the trivialization of $\mathbb{S}|_U$. According to [26], there exists a smooth fiberwise pairing on $\mathbb{S}|_U$ satisfying Property (ii), and any two such pairings differ by multiplication of a smooth function $f \in C^\infty(U)$. By Eq. (3.21), this function f must be a constant function.

As Eq. (3.21) is independent of choices of connections ∇ and $\nabla^\mathbb{S}$, so is the pseudo-Hermitian pairing $\langle \cdot, \cdot \rangle_U$. This concludes the proof. \square

Since one can integrate 1-densities, a spinor pairing yields a pseudo-Hermitian scalar product on the space of compactly supported sections $\Gamma_c(\mathbb{S}|_U)$:

$$(3.22) \quad (\phi, \psi)_U = \int_U \langle \phi, \psi \rangle_U, \quad \forall \phi, \psi \in \Gamma_c(\mathbb{S}|_U).$$

This is called a *spinor scalar product*. It is not clear if $(\cdot, \cdot)_U$ can be extended to a global scalar product on M . However, as we see below, the spinor scalar products over $\mathbb{S}|_U$ enable us to introduce a globally defined adjoint operation on $\mathcal{D}(M, \mathbb{S})$.

Lemma 3.15. *For any $D \in \mathcal{D}(M, \mathbb{S})$, there exists a unique differential operator $D^* \in \mathcal{D}(M, \mathbb{S})$ such that*

$$(3.23) \quad (D\phi, \psi)_U = (\phi, D^*\psi)_U, \quad \forall \phi, \psi \in \Gamma_c(\mathbb{S}|_U),$$

where U is any contractible open subset of M and $(\cdot, \cdot)_U$ is any spinor scalar product on $\mathbb{S}|_U$.

Proof. Let $D \in \mathcal{D}(M, \mathbb{S})$. Denote by $U \subset M$ a contractible open subset and by $D|_U \in \mathcal{D}(U, \mathbb{S}|_U)$ the restriction of the operator D to U . Choose a spinor scalar product $(\cdot, \cdot)_U$. As is well-known, there exists a unique operator $(D|_U)^* \in \mathcal{D}(U, \mathbb{S}|_U)$ satisfying Eq. (3.23). Clearly, this operator satisfies also Eq. (3.23) for the spinor scalar product $a(\cdot, \cdot)_U$, with $a \in \mathbb{R}^\times$. By Proposition 3.14, $(D|_U)^*$ satisfies then Eq. (3.23) for any choice of spinor scalar products on $\mathbb{S}|_U$.

Since the operators $(D|_U)^*$ are uniquely defined by Property (3.23), they glue together into a globally defined differential operator $D^* \in \mathcal{D}(M, \mathbb{S})$. \square

The map $D \mapsto D^*$ is called the *adjoint operation*, and admits the usual properties of the standard adjoint operation.

Proposition 3.16. *The adjoint operation $D \mapsto D^*$ satisfies the following properties:*

- (1) *it is an involutive map on $\mathcal{D}(M, \mathbb{S})$ preserving the filtration;*
- (2) *it is a \mathbb{C} -antilinear antiautomorphism:*

$$(3.24) \quad \begin{cases} (\lambda_1 D_1 + \lambda_2 D_2)^* = \overline{\lambda_1} D_1^* + \overline{\lambda_2} D_2^*, \\ (D_1 D_2)^* = D_2^* D_1^*, \end{cases} \quad \forall \lambda_1, \lambda_2 \in \mathbb{C}, D_1, D_2 \in \mathcal{D}(M, \mathbb{S}),$$

uniquely determined by the following relations

$$(3.25) \quad f^* = \overline{f}, \quad \gamma(\xi)^* = \gamma(\xi) \quad \text{and} \quad (\nabla_X^{\mathbb{S}})^* = -\nabla_X^{\mathbb{S}} - \text{Tr } \nabla X,$$

*for all $f \in C^\infty(M) \otimes \mathbb{C}$, $\xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. Here, $\text{Tr } \nabla X \in C^\infty(M)$ is the trace of $\nabla X \in \Gamma(T^*M \otimes TM)$;*

- (3) *it does not depend on the choice of connections ∇ on TM and $\nabla^{\mathbb{S}}$ on \mathbb{S} .*

Proof. It suffices to prove all the properties above over a contractible open subset $U \subset M$. Choose a spinor scalar product $(\cdot, \cdot)_U$. The formula (3.23) characterizes the map $*$. As a consequence, Eq. (3.24) holds as well as the relations $(D^*)^* = D$, $f^* = \overline{f}$. Using in addition Proposition 3.14, we obtain that $\gamma(\xi)^* = \gamma(\xi)$ and that $*$ is independent from any choice of connections. The equation $(\nabla_X^{\mathbb{S}})^* = -\nabla_X^{\mathbb{S}} - \text{Tr } \nabla X$ follows then from the identities $L_X v = \nabla_X v + (\text{Tr } \nabla X)v$ and $\int_U L_X v = 0$, $\forall v \in \Gamma(|\wedge^{\text{top}} T^*M|)$. Since $\mathcal{D}(M, \mathbb{S}|_U)$ is generated by $C^\infty(U)$, $\Gamma(E|_U)$ and $\Gamma(TU)$, Eqns (3.24)-(3.25) completely determine the adjoint operation $*$. Finally, one easily see that if $D \in \mathcal{D}_k(M, \mathbb{S}|_U)$ then $D^* \in \mathcal{D}_k(M, \mathbb{S}|_U)$. This concludes the proof of the proposition. \square

Denote by $\bar{\cdot}$ the complex conjugation on \mathbb{C} and its natural extension to any complexified \mathbb{R} -vector space $V \otimes \mathbb{C}$. In what follows, we will extend this operation to the algebra $\mathcal{D}(M, \mathbb{S})$.

Proposition 3.17. *There exists a unique \mathbb{C} -antilinear algebra morphism*

$$\bar{\cdot} : \mathcal{D}(M, \mathbb{S}) \longrightarrow \mathcal{D}(M, \mathbb{S}),$$

which coincides with the complex conjugation on $\mathcal{D}_0(M, \mathbb{S}) \cong C^\infty(M) \otimes \mathbb{C}$ and satisfies the following properties:

$$(3.26) \quad \overline{\gamma(\xi)} = \gamma(\bar{\xi}) \quad \text{and} \quad \overline{\nabla_X^{\mathbb{S}}} = \nabla_X^{\mathbb{S}}, \quad \forall \xi \in \Gamma(E) \otimes \mathbb{C}, X \in \mathfrak{X}(M) \otimes \mathbb{C},$$

for any induced spinor connection $\nabla^{\mathbb{S}}$.

Proof. Choose a spinor connection on S and an affine connection on TM . Let $\nabla^{\mathbb{S}}$ be the induced connection on \mathbb{S} . Since $\mathcal{D}(M, \mathbb{S})$ is generated by f , $\gamma(\xi)$, $\nabla_X^{\mathbb{S}}$, with $f \in C^\infty(M) \otimes \mathbb{C}$, $\xi \in \Gamma(E)$, $X \in \mathfrak{X}(M)$, a \mathbb{C} -antilinear algebra morphism $\bar{\cdot}$ satisfying Eq. (3.26) must be unique if it exists. Hence, it suffices to prove the existence of $\bar{\cdot}$ locally. Over a contractible open subset of M , we extend the conjugation of complex functions by setting

$$\overline{f\gamma(\eta)\nabla_\alpha^{\mathbb{S}}} := \bar{f}\gamma(\eta)\nabla_\alpha^{\mathbb{S}},$$

with $f \in C^\infty(M) \otimes \mathbb{C}$, $\eta \in \Gamma(\wedge E)$ and $\alpha = (\alpha_1, \dots, \alpha_k)$ an ordered multi-index. By Lemma 3.6, the map $\bar{\cdot}$ is well-defined. It is obvious to see that $\bar{\cdot}$ is indeed a \mathbb{C} -antilinear involutive map satisfying Eq. (3.26). To check that $\bar{\cdot}$ is in addition an algebra morphism, one may use the commutation relations (3.13), and note that the curvature 2-form of $\nabla^{\mathbb{S}}$ is valued in the real Clifford bundle $\text{Cl}(E)$ (see Eq. (3.20)).

According to Eq. (3.20), different choices of connections on TM and S induce connections on \mathbb{S} which differ by a real term. Therefore the map $\bar{\cdot}$ is independent of choices of connections. This concludes the proof. \square

The map $\bar{\cdot}$ introduced above is called the *conjugation map*. By $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$, we denote the real subalgebra of $\mathcal{D}(M, \mathbb{S})$ generated by sections of the real Clifford bundle $\text{Cl}(E)$ and the covariant derivatives $\nabla_X^{\mathbb{S}}$, with $X \in \mathfrak{X}(M)$. The next proposition gives an intrinsic description of this real subalgebra.

Proposition 3.18. *The real subalgebra $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$ is the fixed point set of $\bar{\cdot}$. Therefore we have $\mathcal{D}(M, \mathbb{S}) \cong \mathcal{D}(M, \mathbb{S})_{\mathbb{R}} \otimes \mathbb{C}$.*

Proof. The proof is straightforward and is left for the reader. \square

Remark 3.19. Assume there exists a real spinor bundle $\mathbb{S}_{\mathbb{R}}$ such that $\mathbb{S} \cong \mathbb{S}_{\mathbb{R}} \otimes \mathbb{C}$. Then, the conjugation map on $\mathcal{D}(M, \mathbb{S}_{\mathbb{R}} \otimes \mathbb{C})$ is induced by the natural conjugation on $\mathbb{S}_{\mathbb{R}} \otimes \mathbb{C}$, so that $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}} \cong \mathcal{D}(M, \mathbb{S}_{\mathbb{R}})$. However, $\mathbb{S}_{\mathbb{R}}$ may not exist.

The following result is obvious.

Proposition 3.20. *For any $k, l \in \mathbb{N}$, let $m = 2k + l$. Assume that the principal symbol of $D \in \mathcal{D}_m(M, \mathbb{S})$ satisfies $\sigma_m(D) \in \mathcal{O}_k(T^*[2]M) \otimes_{C^\infty} \mathcal{O}_l^{\mathbb{C}}(E[1])$. Then we have*

$$\sigma_m(\overline{D}) = \overline{\sigma_m(D)} \quad \text{and} \quad \sigma_m(D^*) = (-1)^k (-1)^{\frac{l(l-1)}{2}} \overline{(\sigma_m(D))}.$$

4. QUANTIZATION OF SYMPLECTIC GRADED MANIFOLDS OF DEGREE 2

First, we recall some basic materials on the Weyl quantization on $T^*\mathbb{R}^n$ and its extension to arbitrary cotangent bundles T^*M [35, 37]. Then, we introduce a similar construction of quantizations on symplectic graded manifolds of degree 2.

4.1. Weyl quantization on $T^*\mathbb{R}^n$. Let $V \rightarrow \mathbb{R}^n$ be a complex vector bundle over \mathbb{R}^n , and denote by $\mathcal{D}(\mathbb{R}^n, V)$ the algebra of differential operators on V . Choosing a trivialization of V , one readily gets that $\mathcal{D}(\mathbb{R}^n, V) \cong \mathcal{D}(\mathbb{R}^n) \otimes_{C^\infty} \Gamma(\text{End } V)$. Recall that the normal order quantization is defined, in terms of the canonical coordinate system (x^i, p_i) on $T^*[2]\mathbb{R}^n$, by

$$\begin{aligned} \mathcal{N}_h : \mathcal{O}(T^*[2]\mathbb{R}^n) \otimes_{C^\infty} \Gamma(\text{End } V) &\longrightarrow \mathcal{D}(\mathbb{R}^n, V) \\ F^{i_1 \cdots i_k}(x) p_{i_1} \cdots p_{i_k} &\longmapsto F^{i_1 \cdots i_k}(x) \left(\frac{\hbar}{i} \frac{\partial}{\partial x^{i_1}} \right) \cdots \left(\frac{\hbar}{i} \frac{\partial}{\partial x^{i_k}} \right), \end{aligned}$$

where $F^{i_1 \cdots i_k} \in \Gamma(\text{End } V)$ and $\hbar \in \mathbb{C}^\times$ is a parameter². Let $\text{Div} = \frac{\partial^2}{\partial x^i \partial p_i}$ be the divergence-like operator acting on the symbol algebra $\mathcal{O}(T^*[2]\mathbb{R}^n)$.

Definition 4.1. *On $T^*[2]\mathbb{R}^n$, the $\mathcal{D}(\mathbb{R}^n, V)$ -valued Weyl quantization is a \mathbb{C} -linear isomorphism, $\mathcal{Q}_h^{\mathbb{R}^n} : \mathcal{O}(T^*[2]\mathbb{R}^n) \otimes_{C^\infty} \Gamma(\text{End } V) \rightarrow \mathcal{D}(\mathbb{R}^n, V)$, indexed by $\hbar \in \mathbb{C}^\times$ and defined by*

$$(4.1) \quad \mathcal{Q}_h^{\mathbb{R}^n} := \mathcal{N}_h \circ \exp \left(\frac{\hbar}{2i} \text{Div} \otimes \text{id} \right).$$

In particular, the Weyl quantization satisfies $\mathcal{Q}_h^{\mathbb{R}^n}(p_i) = \frac{\hbar}{i} \frac{\partial}{\partial x^i}$ and $\mathcal{Q}_h^{\mathbb{R}^n}(F) = F$, for any $F \in \Gamma(\text{End } V)$. In the classical case when V is a trivial line bundle, $\Gamma(\text{End } V)$ is identified with $C^\infty(\mathbb{R}^n)$, and $\mathcal{Q}_h^{\mathbb{R}^n}(F)$ is the multiplication by F . For polynomials in the coordinates (x^i, p_i) , the Weyl quantization is just the symmetrization map satisfying $\mathcal{Q}_h^{\mathbb{R}^n}(p_i) = \frac{\hbar}{i} \frac{\partial}{\partial x^i}$ and $\mathcal{Q}_h^{\mathbb{R}^n}(x^i) = x^i$.

The Weyl quantization can also be defined by the integral formula (4.2) below [12].

Proposition 4.2. *The Weyl quantization satisfies*

$$(4.2) \quad (\mathcal{Q}_h^{\mathbb{R}^n}(F) \psi)(x) = \frac{1}{(2\pi\hbar)^n} \int_{T_x^*\mathbb{R}^n \oplus T_x\mathbb{R}^n} e^{-\frac{i}{\hbar} \langle p, v \rangle} F(x + v/2, p) \cdot \psi(x + v) \, dp \, dv,$$

for all $\psi \in \Gamma(V)$ and $F \in \mathcal{O}(T^*[2]\mathbb{R}^n) \otimes_{C^\infty} \Gamma(\text{End } V)$.

²In this paper, we use the notation \hbar to denote a nonzero variable in \mathbb{C} . In most situation, \hbar can be interpreted as the Planck constant. However, we sometimes let $\hbar = i$.

4.2. Exponential map and parallel transport. To generalize the above Weyl quantization to arbitrary manifolds and vector bundles we will need connections. We fix notations concerning exponential map and parallel transport below.

An affine connection ∇ on TM induces an exponential map, indexed by $x \in M$,

$$\exp_x : U_x \longrightarrow M,$$

where U_x is an open neighborhood of $0 \in T_x M$. We choose U_x such that \exp_x is a diffeomorphism onto its image. In addition, let ∇^V be a connection on a vector bundle $V \rightarrow M$. Then we have parallel transport maps,

$$\mathcal{T}_{x,y}^V : V_x \longrightarrow V_y,$$

for all pairs of points $x, y \in M$ such that $y = \exp_x v$ with $v \in U_x$. Exponential map and parallel transport together induce a local isomorphism of vector bundles

$$(4.3) \quad \begin{aligned} \mathcal{T}_x : U_x \times V_x &\longrightarrow V \\ (v, \phi) &\longmapsto (y, \mathcal{T}_{x,y}^V \phi), \end{aligned}$$

where $y = \exp_x v$. Consider a cut-off function $\chi \in C^\infty(TM)$, i.e., a function equal to 1 in a neighborhood of the zero section of TM such that the support of $\chi(x, \cdot)$ is included in U_x . Define a map

$$(4.4) \quad \chi(x, \cdot) \mathcal{T}_x^* : \Gamma(V) \longrightarrow C^\infty(T_x M) \otimes_{\mathbb{R}} V_x,$$

by setting

$$(\chi(x, \cdot) \mathcal{T}_x^* \psi)(v) = \chi(x, v) (\mathcal{T}_x^* \psi)(v), \quad \forall v \in T_x M, \psi \in \Gamma(V).$$

Note that by definition, if $v \in U_x$ and $y = \exp_x v$, we have

$$(4.5) \quad (\mathcal{T}_x^* \psi)(v) = \mathcal{T}_{y,x}^V (\psi(y)).$$

The connections ∇ and ∇^V induce a connection on the vector bundle $STM \otimes \text{End } V$, and therefore we have a map analogous to (4.4). Since $\mathcal{O}(T^*[2]M) \cong \Gamma(STM)$, this map reads as

$$\chi(x, \cdot) \mathcal{T}_x^* : \mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{End } V) \longrightarrow \mathcal{O}(T^*[2](T_x M)) \otimes_{\mathbb{R}} \text{End}(V_x).$$

If $(v, p) \in T^*U_x$ and $y = \exp_x v$, we have the following explicit formula

$$(4.6) \quad (\mathcal{T}_x^* F)(v, p) := \mathcal{T}_{y,x}^V \circ F(y, \mathcal{T}_{x,y} p) \circ \mathcal{T}_{x,y}^V,$$

where $\mathcal{T}_{x,y} : T_x^* M \rightarrow T_y^* M$ is the parallel transport map induced by ∇ .

4.3. Weyl quantization on $T^*[2]M$. The integral formula (4.2) defining Weyl quantization has been extended to arbitrary cotangent bundles T^*M with the help of a connection on TM [35]. It has been further generalized to differential operators acting on any vector bundle $V \rightarrow M$, using an additional connection on V [37]. We recall such a construction.

Definition 4.3. *By a Weyl quantization map, we mean a map $\mathcal{Q}_h^M : \mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{End } V) \rightarrow \text{End}(\Gamma(V))$ defined by*

$$(4.7) \quad (\mathcal{Q}_h^M(F) \psi)(x) = \frac{1}{(2\pi\hbar)^n} \int_{T_x^* M \oplus T_x M} e^{-\frac{i}{\hbar} \langle p, v \rangle} (\mathcal{T}_x^* F)(v/2, p) \cdot (\mathcal{T}_x^* \psi)(v) \chi(x, v) dp dv,$$

for all $x \in M$, $\psi \in \Gamma(V)$ and $F \in \mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{End } V)$.

The map \mathcal{Q}_h^M depends on the choice of connections on TM and V through the pull-backs $\mathcal{T}_x^* \psi$ and $\mathcal{T}_x^* F$. Note that a priori, \mathcal{Q}_h^M depends also on the cut-off function $\chi \in C^\infty(TM)$.

For any $x \in M$, the projection $\mathbb{R}^n \times V_x \rightarrow \mathbb{R}^n$ defines a trivial vector bundle on \mathbb{R}^n . Denote by

$$\mathcal{Q}_h^{\mathbb{R}^n} : \mathcal{O}(T^*[2]\mathbb{R}^n) \otimes_{\mathbb{R}} \Gamma(\text{End } V_x) \longrightarrow \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n \times V_x)$$

the corresponding Weyl quantization map, given by Eq. (4.1) or equivalently Eq. (4.2). The maps $\mathcal{Q}_h^{\mathbb{R}^n}$ and \mathcal{Q}_h^M are related by parallel transportation as indicated below.

Lemma 4.4. *For any $x \in M$, $\psi \in \Gamma(V)$ and $F \in \mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{End } V)$, we have*

$$\mathcal{T}_x^* \left(\mathcal{Q}_h^M(F) \psi \right) (w) = \left(\mathcal{Q}_h^{\mathbb{R}^n}(\mathcal{T}_x^* F) \mathcal{T}_x^* \psi \right) (w), \quad \forall w \in U_x,$$

where U_x an open neighborhood of $0 \in T_x M$, on which \exp_x is a diffeomorphism onto its image.

Proof. For all $w \in U_x$, Eqns (4.5) and (4.7) imply that

$$(4.8) \quad \mathcal{T}_x^* \left(\mathcal{Q}_h^M(F) \psi \right) (w) = \frac{1}{(2\pi\hbar)^n} \int_{T_y^* M \oplus T_y M} e^{-\frac{i}{\hbar} \langle p, v \rangle} \mathcal{T}_{y,x}^V \left[(\mathcal{T}_y^* F)(v/2, p) \cdot (\mathcal{T}_y^* \psi)(v) \right] \chi(y, v) dp dv,$$

where $y = \exp_x w$. The equality $\exp_y v = \exp_x(w + \mathcal{T}_{y,x} v)$ and Eq. (4.5) lead to

$$(4.9) \quad (\mathcal{T}_y^* \psi)(v) = \mathcal{T}_{x,y}^V [(\mathcal{T}_x^* \psi)(w + \mathcal{T}_{y,x} v)].$$

Similarly, using $\exp_y \frac{v}{2} = \exp_x(w + \frac{1}{2} \mathcal{T}_{y,x} v)$ and Eq. (4.6), we have

$$(4.10) \quad \mathcal{T}_{y,x}^V \circ \left[(\mathcal{T}_y^* F)(v/2, p) \right] = \left[(\mathcal{T}_x^* F)(w + \frac{1}{2} \mathcal{T}_{y,x} v, \mathcal{T}_{y,x} p) \right] \circ \mathcal{T}_{y,x}^V.$$

As parallel transport in the fibers of $TM \oplus T^*M$ preserves the duality pairing $\langle p, v \rangle$ and the measure $dp dv$, by Eqns (4.8)-(4.10) and the change of variables $(\mathcal{T}_{y,x} v, \mathcal{T}_{y,x} p) \mapsto (v, p)$, we obtain

$$\begin{aligned} \mathcal{T}_x^* \left(\mathcal{Q}_h^M(F) \psi \right) (w) &= \frac{1}{(2\pi\hbar)^n} \int_{T_x^* M \oplus T_x M} e^{-\frac{i}{\hbar} \langle p, v \rangle} (\mathcal{T}_x^* F)(w + v/2, p) \cdot (\mathcal{T}_x^* \psi)(w + v) \\ &\quad \chi(\exp_x w, \mathcal{T}_{x, \exp_x w} v) dp dv. \end{aligned}$$

Since F is polynomial in p , its Fourier transform w.r.t. p is a distribution supported at 0. As the function $v \mapsto \chi(x, v)$ is equal to 1 in a neighborhood of the zero section, the right hand side in the above equation reduces to $(\mathcal{Q}_h^{\mathbb{R}^n}(\mathcal{T}_x^* F) \mathcal{T}_x^* \psi)(w)$ (see formula (4.2)). The conclusion follows. \square

As a straightforward consequence, we get

Proposition 4.5. *For any pair of connections on TM and V , there is a unique Weyl quantization map, $\mathcal{Q}_h^M : \mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{End } V) \rightarrow \mathcal{D}(M, V)$. Moreover, \mathcal{Q}_h^M is a linear isomorphism.*

In what follows, we assume a choice of connections on TM and V is made and we refer to the corresponding map \mathcal{Q}_h^M as the Weyl quantization map.

Let V be a vector bundle endowed with a pseudo-Hermitian pairing $\langle \cdot, \cdot \rangle_V : \Gamma(V) \times \Gamma(V) \rightarrow \Gamma(|\wedge^{\text{top}} T^*M|) \otimes \mathbb{C}$ as in Eq. (3.19). Then, the formula, $(\phi, \psi) := \int_M \langle \phi, \psi \rangle_V$ defines a pseudo-Hermitian scalar product on $\Gamma_c(V)$. Denote by $*$: $\mathcal{D}(M, V) \rightarrow \mathcal{D}(M, V)$ the adjoint operation associated to (\cdot, \cdot) and by $*_V : \Gamma(\text{End } V) \rightarrow \Gamma(\text{End } V)$ the adjoint operation associated to $\langle \cdot, \cdot \rangle_V$.

Proposition 4.6. *Assume that the following equality holds*

$$(4.11) \quad \nabla_X \langle \phi, \psi \rangle_V = \langle \nabla_X^V \phi, \psi \rangle_V + \langle \phi, \nabla_X^V \psi \rangle_V, \quad \forall X \in \mathfrak{X}(M), \phi, \psi \in \Gamma(V).$$

Then, the Weyl quantization \mathcal{Q}_h^M satisfies the property

$$\mathcal{Q}_h^M(F)^* = \mathcal{Q}_h^M(F^{*V}),$$

for all $F \in \mathcal{O}(T^[2]M) \otimes_{C^\infty} \Gamma(\text{End } V)$.*

Proof. Eqns (4.5)–(4.7) yield the formula

$$(\mathcal{Q}_h^M(F)\psi)(x) = \frac{1}{(2\pi\hbar)^n} \int_{T_x^*M \oplus T_x M} e^{-\frac{i}{\hbar}\langle p, v \rangle} \mathcal{T}_{y,x}^V \left[F(y, \mathcal{T}_{x,y}p) \cdot \mathcal{T}_{z,y}^V(\psi(z)) \right] \chi(x, v) \, dp \, dv,$$

where $y = \exp_x(v/2)$ and $z = \exp_x(v)$. By Eq. (4.11), the parallel transports \mathcal{T} and \mathcal{T}^V satisfy the relation

$$\mathcal{T}_{y,x}(\langle \phi(y), \psi(y) \rangle_V) = \left\langle \mathcal{T}_{y,x}^V(\phi(y)), \mathcal{T}_{y,x}^V(\psi(y)) \right\rangle_V, \quad \forall \phi, \psi \in \Gamma(V),$$

which implies

$$\left\langle \mathcal{T}_{y,x}^V \left[F(y, \mathcal{T}_{x,y}p)^{*V} \cdot \mathcal{T}_{z,y}^V(\phi(z)) \right], \psi(x) \right\rangle_V = \mathcal{T}_{z,x} \left\langle \phi(z), \mathcal{T}_{y,z}^V \left[F(y, \mathcal{T}_{x,y}p) \cdot \mathcal{T}_{x,y}^V(\psi(x)) \right] \right\rangle_V.$$

We deduce that

$$(4.12) \quad (\mathcal{Q}_h^M(F^{*V})\phi, \psi) = \frac{1}{(2\pi\hbar)^n} \times \int_M \int_{T_x M \oplus T_x^* M} \mathcal{T}_{z,x} \left\langle \phi(z), \mathcal{T}_{y,z}^V \left[F(y, \mathcal{T}_{x,y}p) \cdot \mathcal{T}_{x,y}^V(\psi(x)) \right] \right\rangle_V e^{\frac{i}{\hbar}\langle p, v \rangle} \chi(x, v) \, dp \, dv.$$

Let us recall that, for integration of 1-densities on $TM \oplus T^*M$, the change of variables formula reads as

$$\int_{TM \oplus T^*M} (\Phi^*G)(X) = \int_{TM \oplus T^*M} \mathcal{T}_{Z,X} (G \circ \Phi(X)) = \int_{TM \oplus T^*M} G(Z),$$

where $\Phi : X \mapsto Z$ is a diffeomorphism and G a 1-density. We apply the above formula to Eq. (4.12) with the change of variables $(x, v, p) \mapsto (z, v', p')$, where $z = \exp_x v$, $v' = -\mathcal{T}_{x,z}v$ and $p' = \mathcal{T}_{x,z}p$. Since $dp \, dv = \mathcal{T}_{z,x}(dp' \, dv')$ and $\langle p, v \rangle = \langle p', v' \rangle$, we obtain

$$(\mathcal{Q}_h^M(F^{*V})\phi, \psi) = \frac{1}{(2\pi\hbar)^n} \times \int_M \left\langle \phi(z), \int_{T_z M \oplus T_z^* M} \mathcal{T}_{y,z}^V \left[F(y, \mathcal{T}_{z,y}p') \cdot \mathcal{T}_{x,y}^V(\psi(x)) \right] e^{-\frac{i}{\hbar}\langle p', v' \rangle} \chi(x, \mathcal{T}_{z,x}v') \, dp' \, dv' \right\rangle_V,$$

where $y = \exp_z(v'/2)$ and $x = \exp_z(v')$. Using Eq. (4.7), we conclude that $(\mathcal{Q}_h^M(F^{*v})\phi, \psi) = (\phi, \mathcal{Q}_h^M(F)\psi)$, $\forall \phi, \psi \in \Gamma(V)$, and the result follows. \square

Finally, let us describe an explicit formula for the Weyl quantization \mathcal{Q}_h^M in low degrees. By abuse of notation, we also denote by ∇ the induced connection on $TM \otimes \text{End } V$. For a vector bundle $W \rightarrow M$, let

$$(4.13) \quad \text{Tr} : \Gamma(W \otimes TM \otimes T^*M) \longrightarrow \Gamma(W),$$

be the trace map.

Proposition 4.7. *The Weyl quantization \mathcal{Q}_h^M satisfies*

$$(4.14) \quad \mathcal{Q}_h^M(F) = F \quad \text{and} \quad \mathcal{Q}_h^M(X \otimes F) = \frac{\hbar}{i} \left[F \nabla_X^V + \frac{1}{2} \text{Tr } \nabla(X \otimes F) \right],$$

for all $F \in \Gamma(\text{End } V)$ and $X \in \mathcal{O}_2(T^*[2]M) \cong \mathfrak{X}(M)$.

Proof. The first case is trivial. For second equation, choose an open neighborhood U_x of $0 \in T_x M$ such that $\exp_x : U_x \rightarrow \exp_x(U_x)$ is a diffeomorphism. By pull-back by \mathcal{T}_x^{-1} of a Cartesian coordinate system on $T_x M \times V_x$, we obtain normal coordinates (x^i) centered at x and a trivialization of V over $\exp_x(U_x) \subset M$. Denote by (x^i, p_i) the induced canonical coordinates on $T_x^*(\exp_x(U_x))$. Under such coordinates, we can write $X(x) = X^i(x)p_i$, with $X^i \in C^\infty(\exp_x(U_x))$. By Lemma 4.4 and Eq. (4.1), we have

$$(4.15) \quad \left(\mathcal{Q}_h^M(X \otimes F) \psi \right)(x) = \left(F(x) X^i(x) \frac{\hbar}{i} \frac{\partial}{\partial x^i} + \frac{\hbar}{2i} \frac{\partial}{\partial x^i} \left(F(x) X^i(x) \right) \right) \psi(x), \quad \forall \psi \in \Gamma(V).$$

Since (x^i) are normal coordinates at x , the partial derivatives $\frac{\partial}{\partial x^i}$ coincide with covariant derivatives at the point x . Hence, Eq. (4.14) holds at x . The point x being arbitrary, this implies the result. \square

4.4. Weyl quantization on symplectic graded manifolds of degree 2. Let (E, g) be a pseudo-Euclidean vector bundle equipped with a metric connection ∇^E . Assume that (E, g) admits a spinor bundle S . Choose an affine connection ∇ on TM , and choose a spinor connection on S , compatible with ∇^E (see Eq. (3.4)). They induce a Weyl quantization \mathcal{Q}_h^M , valued in $\mathcal{D}(M, S)$.

Definition 4.8. *The Weyl quantization \mathcal{WQ}_h on the symplectic graded manifold $(T^*[2]M \oplus E[1], \omega_{g, \nabla^E})$ is defined by the following commutative diagram*

$$(4.16) \quad \begin{array}{ccc} \mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1]) & \xrightarrow{\mathcal{WQ}_h} & \mathcal{D}(M, S) \\ \gamma_h \downarrow & \nearrow \mathcal{Q}_h^M & \\ \mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{End } S) & & \end{array}$$

The vertical map reads as

$$(4.17) \quad \gamma_h := \text{id} \otimes \left(\frac{\hbar}{2i} \right)^{\kappa/2} \gamma, \quad \text{on } \mathcal{O}^{\mathbb{C}}(T^*[2]M) \otimes_{C^\infty} \Gamma(\wedge^\kappa E),$$

where γ is the standard Clifford quantization map (see (3.1)).

The Weyl quantization \mathcal{WQ}_h extends those studied in [13, 36], with E being TM and $TM \oplus T^*M$ respectively.

Theorem 4.9. *For all $F \in \mathcal{O}_k^{\mathbb{C}}(T^*[2]M \oplus E[1])$, $k \in \mathbb{N}$, the Weyl quantization \mathcal{WQ}_h satisfies the following properties:*

- (i) $\mathcal{WQ}_h(F) = (\hbar/i)^{k/2} \mathcal{WQ}_i(F)$;
- (ii) $\sigma_k \circ \mathcal{WQ}_h(F) = (\hbar/i)^{k/2} F$ (see Definition 3.10 of the principal symbol map σ_k);
- (iii) $\mathcal{WQ}_h(F) \in \mathcal{D}^+(M, S)$ if k is even and $\mathcal{WQ}_h(F) \in \mathcal{D}^-(M, S)$ if k is odd (see Eq. (3.6));
- (iv) $[\mathcal{WQ}_h(F), \mathcal{WQ}_h(G)] = \frac{\hbar}{i} \mathcal{WQ}_h(\{F, G\}) + O(\hbar^2)$, for all $G \in \mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1])$.

Proof. As $\mathcal{O}_k^{\mathbb{C}}(T^*[2]M \oplus E[1]) = \bigoplus_{2\ell+\kappa=k} \mathcal{O}_\ell(T^*[2]M) \otimes_{C^\infty} \mathcal{O}_\kappa(E[1])$, it suffices to check Properties (i), (ii) and (iii) on homogeneous functions $F \in \mathcal{O}_\ell(T^*[2]M) \otimes_{C^\infty} \mathcal{O}_\kappa(E[1])$. This is easy using Lemma 4.4 and Eqns (4.1), (4.17).

Let $F \in \mathcal{O}_k^{\mathbb{C}}(T^*[2]M \oplus E[1])$ and $G \in \mathcal{O}_l^{\mathbb{C}}(T^*[2]M \oplus E[1])$. Set

$$H = \mathcal{WQ}_i^{-1}([\mathcal{WQ}_i(F), \mathcal{WQ}_i(G)] - \mathcal{WQ}_i(\{F, G\})).$$

Using Proposition 3.11 together with (ii) and (iii), we deduce that H has degree $k + l - 4$. The claim (iv) follows then from (i). \square

We provide an explicit expression for the Weyl quantization \mathcal{WQ}_h in low degrees. By abuse of notation, we denote by ∇ the connection on $\wedge E \otimes TM$ induced by the connections on E and TM . From the definition of \mathcal{WQ}_h and Proposition 4.7, we deduce

Proposition 4.10. *The Weyl quantization \mathcal{WQ}_h satisfies*

$$(4.18) \quad \begin{aligned} \mathcal{WQ}_h(F) &= \left(\frac{\hbar}{2i}\right)^{\kappa/2} \gamma(F), \\ \mathcal{WQ}_h(FX) &= \left(\frac{\hbar}{2i}\right)^{\kappa/2} \left(\frac{\hbar}{i}\right)^k \left[\gamma(F) \nabla_X^S + \frac{1}{2} \gamma(\text{Tr } \nabla(XF)) \right], \end{aligned}$$

for all $F \in \mathcal{O}_\kappa^{\mathbb{C}}(E[1])$ and $X \in \mathcal{O}_2(T^*[2]M) \cong \mathfrak{X}(M)$. Here, the trace map Tr is defined as in (4.13), with $W = \wedge^\kappa E \otimes \mathbb{C}$.

To go further, we assume that (E, g) admits a twisted spinor bundle $\mathbb{S} := S \otimes (\det S^*)^{1/N} \otimes |\wedge^{\text{top}} T^*M|^{1/2}$, with N being the rank of S . An affine connection ∇ on TM and a spinor connection on S induce a spinor connection $\nabla^{\mathbb{S}}$ on \mathbb{S} and a Weyl quantization map

$$(4.19) \quad \mathcal{WQ}_h : \mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1]) \longrightarrow \mathcal{D}(M, \mathbb{S}).$$

The latter satisfies in particular Theorem 4.9 and Proposition 4.10. Obviously, one should replace ∇^S by $\nabla^{\mathbb{S}}$ in Eq. (4.18). Considering \mathbb{S} allows for further structures on $\mathcal{D}(M, \mathbb{S})$ (see Section 3.3). We study below their interplay with the map \mathcal{WQ}_h . For all $f \in C^\infty(M)$, $\xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, let

$$\tau(f) = f, \quad \tau(\xi) = i\xi, \quad \tau(X) = X.$$

It is easy to see that τ extends to a \mathbb{C} -antilinear antiautomorphism on $\mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1])$, which is an involution.

Proposition 4.11. *The Weyl quantization map as in Eq. (4.19) satisfies $\mathcal{WQ}_h \circ \tau(\cdot) = \mathcal{WQ}_h(\cdot)^*$, where $*$ is the adjoint operation defined in Lemma 3.15.*

Proof. Given the local nature of the maps \mathcal{WQ}_h , τ and $*$, it is enough to work over a contractible open subset $U \subset M$. Let $\langle \cdot, \cdot \rangle_U$ be a spinor pairing (see Proposition 3.14) and $*_{\mathbb{S}} : \Gamma(\text{End } \mathbb{S}|_U) \rightarrow \Gamma(\text{End } \mathbb{S}|_U)$ its adjoint operation. Then we have $\gamma_h(\xi)^{*_{\mathbb{S}}} = i\gamma_h(\xi) = \gamma_h(\tau(\xi))$. Since both maps $*_{\mathbb{S}}$ and τ are \mathbb{C} -antilinear antiautomorphisms, we get

$$\gamma_h(\tau(F)) = (\gamma_h(F))^{*_{\mathbb{S}}},$$

for all $F \in \mathcal{O}^{\mathbb{C}}(T^*[2]U \oplus E|_U[1])$. By Proposition 3.14, the pair of connections $(\nabla, \nabla^{\mathbb{S}})$ satisfies Eq. (4.11), with $V = \mathbb{S}$. Applying Proposition 4.6 yields the result. \square

The case $\hbar = i$ plays a peculiar role.

Proposition 4.12. *Setting $\mathcal{WQ} := \mathcal{WQ}_i$, we have*

- (1) \mathcal{WQ} restricts to an \mathbb{R} -linear isomorphism $\mathcal{WQ} : \mathcal{O}(T^*[2]M \oplus E[1]) \rightarrow \mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$, where the real algebra $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$ is defined in Proposition 3.18,
- (2) $\mathcal{WQ}(F)^* = (-1)^{\lfloor k/2 \rfloor} \mathcal{WQ}(F)$, for all $F \in \mathcal{O}_k(T^*[2]M \oplus E[1])$, with $k \in \mathbb{N}$ and $\lfloor k/2 \rfloor$ the integer part of k .

Proof. Complex conjugation in \mathbb{C} extends naturally to the algebras $\mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1])$, $\mathcal{O}(T^*[2]M) \otimes_{C^\infty} \Gamma(\text{Cl}(E))$ and $\mathcal{D}(M, \mathbb{S})$ (see Proposition 3.17). To establish (1), it suffices to show that

$$\mathcal{WQ}(\overline{F}) = \overline{\mathcal{WQ}(F)},$$

for all $F \in \mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1])$. Given the local nature of this property, we can restrict ourselves to a contractible open neighborhood U of a point $x \in M$. We have

$$\begin{aligned} \mathcal{WQ}(\overline{F}) &= (\mathcal{T}_x^{-1})^* \circ \mathcal{Q}_i^{\mathbb{R}^n}(\mathcal{T}_x^*(\gamma_i(\overline{F}))) \circ \mathcal{T}_x^*, & \text{by Lemma 4.4,} \\ &= (\mathcal{T}_x^{-1})^* \circ \mathcal{Q}_i^{\mathbb{R}^n}(\mathcal{T}_x^*(\overline{\gamma_i(F)})) \circ \mathcal{T}_x^*, & \text{by equality } \gamma_i = \text{id} \otimes \gamma. \end{aligned}$$

Since \mathcal{T}_x is built from connections on the real vector bundles TM and E , we deduce that $\mathcal{T}_x^*(\overline{\gamma_i(F)}) = \overline{\mathcal{T}_x^*(\gamma_i(F))}$. Therefore, from Eq. (4.1), it follows that

$$\mathcal{Q}_i^{\mathbb{R}^n}(\mathcal{T}_x^*(\overline{\gamma_i(F)})) = \overline{\mathcal{Q}_i^{\mathbb{R}^n}(\mathcal{T}_x^*(\gamma_i(F)))},$$

and finally

$$\mathcal{WQ}(\overline{F}) = (\mathcal{T}_x^{-1})^* \circ [\mathcal{T}_x^* \circ \overline{\mathcal{WQ}(F)} \circ (\mathcal{T}_x^{-1})^*] \circ \mathcal{T}_x^*.$$

From equation above and Proposition 4.10, we deduce that the map $\mathcal{WQ}(F) \mapsto \mathcal{WQ}(\overline{F})$ is an algebra automorphism of $\mathcal{D}(M, \mathbb{S})$ satisfying Eq. (3.26). By Proposition 3.17, this map must be of the form $\mathcal{WQ}(F) \mapsto \overline{\mathcal{WQ}(F)}$. Hence we have $\mathcal{WQ}(\overline{F}) = \overline{\mathcal{WQ}(F)}$.

The second point is a direct consequence of (i) in Theorem 4.9 and Proposition 4.11. \square

The Weyl quantization map \mathcal{WQ}_h as in Eq. (4.19) induces a star-product \star_h on the symplectic manifold $T^*[2]M \oplus E[1]$. The properties of the map \mathcal{WQ}_h , exhibited above, can be rephrased in terms of properties of \star_h .

Corollary 4.13. *The product defined on $\mathcal{O}^{\mathbb{C}}(T^*[2]M \oplus E[1])$ by*

$$F \star_h G := (\mathcal{WQ}_h)^{-1}(\mathcal{WQ}_h(F) \circ \mathcal{WQ}_h(G)),$$

is a symmetric star-product, explicitly given by

$$F \star_h G = \sum_{k=0}^{\infty} \left(\frac{\hbar}{i}\right)^k B_{2k}(F, G),$$

such that, for each $k \in \mathbb{N}$,

- B_{2k} is a real bidifferential operator of degree $-2k$ independent of \hbar ,
- B_{2k} is symmetric if k is even and skew-symmetric if k is odd,
- B_0 is the multiplication and B_2 is the Poisson bracket on $(T^*[2]M \oplus E[1], \omega_{g, \nabla^E})$.

Proof. From Point (i) in Theorem 4.9, we deduce that

$$F \star_h G = \sum_{k=0}^{\infty} \left(\frac{\hbar}{i}\right)^{k/2} B_k(F, G),$$

where the B_k are bilinear operators of degree $-k$. By Lemma 4.4 and Eq. (4.1), they are bidifferential operators. Since \mathcal{WQ}_h preserves parity, we deduce that B_k vanishes if k is odd. The identifications of B_0 and B_2 , with the multiplication and the Poisson bracket respectively, follow from the points (ii) and (iv) in Theorem 4.9.

Set $\hbar = i$. By Proposition 4.12 the operators B_{2k} are real and (skew-)symmetric according to the parity of k . \square

5. APPLICATIONS TO COURANT ALGEBROIDS

5.1. Definition of Courant algebroids. A pre-Courant algebroid is a pseudo-Euclidean vector bundle (E, g) over a smooth manifold M , together with a vector bundle morphism $\rho : E \rightarrow TM$, called the anchor, and an \mathbb{R} -bilinear operation $[\![\cdot, \cdot]\!]$ on $\Gamma(E)$, called the Dorfman bracket, subject to the following rules:

$$\begin{aligned} (5.1) \quad [\![\xi, f \cdot \eta]\!] &= \rho(\xi)[f] \cdot \eta + f \cdot [\![\xi, \eta]\!], \\ [\![\xi, \xi]\!] &= \rho^* d(g(\xi, \xi)), \\ \rho(\xi)[g(\eta, \eta)] &= 2g([\![\xi, \eta]\!], \eta), \end{aligned}$$

for all $f \in C^\infty(M)$ and $\xi, \eta \in \Gamma(E)$. In the second equation above, d stands for the de Rham differential and $\rho^* : T^*M \rightarrow E^* \cong E$ is the dual map of ρ . Moreover, if the bracket satisfies the Jacobi identity

$$(5.2) \quad [\![\xi, [\![\eta_1, \eta_2]\!]]\!] = [[[\xi, \eta_1], \eta_2] + [\![\eta_1, [\![\xi, \eta_2]\!]]\!],$$

for all $\xi, \eta_1, \eta_2 \in \Gamma(E)$, then $(E, g, \rho, [\![\cdot, \cdot]\!])$ is called a Courant algebroid [21, 29].

Example 5.1. For any smooth manifold M , the vector bundle $E = TM \oplus T^*M$ admits a standard Courant algebroid structure, where the anchor is the projection onto the first component and the pairing and Dorfman bracket are given by

$$\begin{aligned} g(X + \alpha, Y + \beta) &= \frac{1}{2}(\langle X, \beta \rangle + \langle Y, \alpha \rangle), \\ \llbracket X + \alpha, Y + \beta \rrbracket &= [X, Y] + L_X \beta - \iota_Y d\alpha, \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M) \cong \Gamma(TM)$ and $\alpha, \beta \in \Gamma(T^*M)$. One can also twist the above bracket by a closed 3-form $H \in \Omega^3(M)$ [33],

$$\llbracket X + \alpha, Y + \beta \rrbracket = [X, Y] + L_X \beta - \iota_Y d\alpha + H(X, Y, \cdot).$$

For more examples, see e.g. [21].

5.2. Courant algebroids and symplectic graded manifolds of degree 2. We assume, from now on, that (E, g) is endowed with a metric connection ∇^E so that the minimal symplectic realization of the Poisson manifold $E[1]$ is given by $(T^*[2]M \oplus E[1], \omega_{g, \nabla^E})$ (see Section 2.3). The Poisson bracket on $T^*[2]M \oplus E[1]$ is denoted by $\{\cdot, \cdot\}$ which is of degree -2 . We will use the following identifications without further comments

$$\begin{aligned} \mathcal{O}_0(T^*[2]M \oplus E[1]) &\cong C^\infty(M), \\ \mathcal{O}_1(T^*[2]M \oplus E[1]) &\cong \Gamma(E), \\ \mathcal{O}_2(T^*[2]M \oplus E[1]) &\cong \Gamma(TM \oplus \wedge^2 E). \end{aligned}$$

Every degree 3 function $\Theta \in \mathcal{O}_3(T^*[2]M \oplus E[1])$ induces a pre-Courant algebroid structure on (E, g) by setting

$$(5.3) \quad \begin{aligned} \rho(\xi_1)[f] &:= \{\{\Theta, \xi_1\}, f\}, \\ \llbracket \xi_1, \xi_2 \rrbracket &:= \{\{\Theta, \xi_1\}, \xi_2\}, \end{aligned} \quad \forall f \in C^\infty(M), \xi_1, \xi_2 \in \Gamma(E).$$

The structural identities (5.1) follow from the fact that $\{\cdot, \cdot\}$ is a Poisson bracket, which satisfies $\{\xi, \eta\} = g(\xi, \eta)$ for all $\xi, \eta \in \Gamma(E)$. Moreover, if $\{\Theta, \Theta\} = 0$, then $\llbracket \cdot, \cdot \rrbracket$ satisfies the Jacobi identity (5.2) and therefore E becomes a Courant algebroid.

Conversely, one can construct a degree 3 function $\Theta \in \mathcal{O}_3(T^*[2]M \oplus E[1])$ out of a pre-Courant algebroid $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ as follows. The anchor map ρ , being a bundle map, can be identified with a section in $\Gamma(TM \otimes E^*)$, which is a function of degree 3 on $T^*[2]M \oplus E[1]$. From g and $\llbracket \cdot, \cdot \rrbracket$, one can define the torsion map $C_\nabla : \Gamma(\wedge^3 E) \rightarrow C^\infty(M)$ by

$$(5.4) \quad C_\nabla(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \text{cycl}_{123} \, g \left(\frac{1}{3} (\llbracket \xi_1, \xi_2 \rrbracket - \llbracket \xi_2, \xi_1 \rrbracket) - \left(\nabla_{\rho(\xi_1)}^E \xi_2 - \nabla_{\rho(\xi_2)}^E \xi_1 \right), \xi_3 \right),$$

where $\xi_1, \xi_2, \xi_3 \in \Gamma(E)$ and cycl_{123} denotes the sum over cyclic permutations. As proven in [1], the identities (5.1) ensure that C_∇ is $C^\infty(M)$ -linear, so that it can be identified with a section in $\Gamma(\wedge^3 E)$. Set $\Theta = \rho + C_\nabla$. Then, Θ is a degree 3 function which satisfies Eq. (5.3). Moreover, if $\llbracket \cdot, \cdot \rrbracket$ satisfies the Jacobi identity (5.2), then we have $\{\Theta, \Theta\} = 0$. Thus we recover the following

Theorem 5.2 ([30]). *Let (E, g) be a pseudo-Euclidean vector bundle over M . There is a bijection between pre-Courant algebroids $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ and degree 3 functions $\Theta \in \mathcal{O}_3(T^*[2]M \oplus E[1])$. They are related via $\Theta = \rho + C_\nabla$.*

Moreover, $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ is a Courant algebroid if and only if $\{\Theta, \Theta\} = 0$.

The above function Θ is called the *Hamiltonian generating function* of the Courant algebroid.

Remark 5.3. Both Θ and the Poisson bracket on $T^*[2]M \oplus E[1]$ depend on the choice of metric connection on E . Via the symplectic diffeomorphism $\Xi_\nabla : \mathcal{M} \rightarrow T^*[2]M \oplus E[1]$, both admit an intrinsic version on \mathcal{M} . They satisfy again Eq. (5.3), see [30].

5.3. Dirac generating operators of Courant algebroids. There is another approach to generate a Courant algebroid, via the so-called *Dirac generating operators* [1, 8].

From now on, we assume that (E, g) admits a spinor bundle S and that $\det(S^*)^{1/N}$ exists, with N being the rank of S . Then, the twisted spinor bundle, $\mathbb{S} := S \otimes (\det S^*)^{1/N} \otimes |\wedge^{\text{top}} T^*M|^{1/2}$, is well-defined. The algebra of real differential operators $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$, defined in Proposition 3.18, inherits a \mathbb{Z}_2 -grading and a filtration from $\mathcal{D}(M, \mathbb{S})$ (cf. Eqns (3.6) and (3.8)). In particular, we have

$$\begin{aligned} \mathcal{D}_0(M, \mathbb{S})_{\mathbb{R}} &\cong C^\infty(M) \cong \mathcal{O}_0(T^*[2]M \oplus E[1]), \\ \mathcal{D}_1^-(M, \mathbb{S})_{\mathbb{R}} &\cong \Gamma(E) \cong \mathcal{O}_1(T^*[2]M \oplus E[1]). \end{aligned}$$

Since the Weyl quantization map $\mathcal{WQ} : \mathcal{O}(T^*[2]M \oplus E[1]) \rightarrow \mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$ (see Proposition 4.12) preserves parity (see Theorem 4.9), it induces the following isomorphism:

$$\mathcal{WQ}^{-1} : \mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}} \xrightarrow{\sim} \mathcal{O}_3(T^*[2]M \oplus E[1]) \oplus \mathcal{O}_1(T^*[2]M \oplus E[1]).$$

The Dorfman bracket can be obtained as a derived bracket of the commutator in $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$, for a well-chosen generating operator $D \in \mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}}$, as shown in [1]. This approach provides a quantum analog to the Hamiltonian picture for Courant algebroids, presented in the previous section. Namely, as the commutator lowers the order by 2 and preserves parity, we can define

$$(5.5) \quad \begin{aligned} \rho(\xi_1)[f] &:= [[D, \gamma(\xi_1)], f] \in C^\infty(M), \\ \llbracket \xi_1, \xi_2 \rrbracket &:= [[D, \gamma(\xi_1)], \gamma(\xi_2)] \in \Gamma(E), \end{aligned} \quad \forall f \in C^\infty(M), \xi_1, \xi_2 \in \Gamma(E).$$

They satisfy the following properties.

Proposition 5.4. *Let $D \in \mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}}$ such that $\sigma_3(D) \neq 0$. Then*

- *the map ρ and the bracket $\llbracket \cdot, \cdot \rrbracket$ given by Eq. (5.5) define a pre-Courant algebroid structure on (E, g) ;*
- *the pre-Courant algebroid defined by D as in Eq. (5.5) coincides with the one defined by its principal symbol $\sigma_3(D)$ as in Eq. (5.3);*
- *the operator D generates a Courant algebroid if and only if $D^2 \in \mathcal{D}_2(M, \mathbb{S})_{\mathbb{R}}$.*

Proof. Using Eq. (3.14), we obtain that the map ρ and the bracket $\llbracket \cdot, \cdot \rrbracket$, defined by D via Eq. (5.5), coincide with the ones determined by $\sigma_3(D)$ via Eq. (5.3). Hence, they define a pre-Courant algebroid structure on (E, g) .

It remains to prove the last point. Since D is odd, we have $D^2 = \frac{1}{2}[D, D]$. As the commutator lowers the order by 2 and preserves the parity, the operator $[D, D]$ is of order 4, 2 or 0. By Eq. (3.14), its principal symbol satisfies $\sigma_4([D, D]) = \{\sigma_3(D), \sigma_3(D)\}$. Hence, $D^2 \in \mathcal{D}_2(M, \mathbb{S})_{\mathbb{R}}$ if and only if $\{\sigma_3(D), \sigma_3(D)\} = 0$. The result follows then from Theorem 5.2. \square

In [1], a stronger condition of Dirac generating operators was introduced.

Definition 5.5. [1] *A Dirac generating operator is an operator $D \in \mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}}$, such that $\sigma_3(D) \neq 0$ and $D^2 \in \mathcal{D}_0(M, \mathbb{S})_{\mathbb{R}}$.*

According to Proposition 5.4, a Dirac generating operator indeed generates a Courant algebroid structure on (E, g) via Eq. (5.5).

The existence of a Dirac generating operator for a given Courant algebroid is nontrivial and was one of the main results of [1]. Moreover, Dirac generating operators are not unique: two Dirac generating operators D, D' defining the same Courant algebroid have the same principal symbol but may differ by an element $\gamma(\xi) \in \mathcal{D}_1^-(M, \mathbb{S})_{\mathbb{R}}$ satisfying $\{\sigma_3(D), \xi\} = 0$. As an application of our theory, we construct a Dirac generating operator D via the Weyl quantization map \mathcal{WQ} . Following an idea of Ševera [32], we uniquely characterize D .

Theorem 5.6. *Let $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ be a Courant algebroid admitting a twisted spinor bundle \mathbb{S} . There exists a unique Dirac generating operator $D \in \mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}}$ satisfying:*

- D generates the given Courant algebroid;
- $D^* = -D$, where $*$ is the adjoint operation defined in (3.23).

Moreover, we have $D = \mathcal{WQ}(\Theta)$, where $\Theta \in \mathcal{O}_3(T^*[2]M \oplus E[1])$ is the Hamiltonian generating function of $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$.

We need a lemma first.

Lemma 5.7. *The Weyl quantization map \mathcal{WQ} induces a linear isomorphism*

$$(5.6) \quad \mathcal{WQ} : \mathcal{O}_3(T^*[2]M \oplus E[1]) \xrightarrow{\sim} \{D \in \mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}} \mid D^* = -D\},$$

which establishes a bijection between Hamiltonian generating functions and skew-symmetric Dirac generating operators.

Proof. According to Proposition 4.12, we have $\mathcal{WQ}(F)^* = (-1)^{\lfloor k/2 \rfloor} \mathcal{WQ}(F)$ for all $F \in \mathcal{O}_k(T^*[2]M \oplus E[1])$. Hence, the map \mathcal{WQ} sends $\mathcal{O}_1(T^*[2]M \oplus E[1])$ to the space of self-adjoint operators in $\mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}}$ and $\mathcal{O}_3(T^*[2]M \oplus E[1])$ to the space of skew-adjoint operators in $\mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}}$. Thus, the map (5.6) is indeed an isomorphism.

Let Θ be a function of degree 3 and $D = \mathcal{WQ}(\Theta)$. By Corollary 4.13, we have

$$2(\mathcal{WQ})^{-1}(D^2) = \{\Theta, \Theta\} + B_4(\Theta, \Theta) + B_6(\Theta, \Theta),$$

where $B_6(\Theta, \Theta)$ is of degree 0 and $B_4(\cdot, \cdot)$ is a symmetric bidifferential operator. As Θ is of odd degree, we get $B_4(\Theta, \Theta) = 0$. As a consequence, we conclude that $\{\Theta, \Theta\} = 0$ is equivalent to $D^2 \in \mathcal{D}_0(M, \mathbb{S})_{\mathbb{R}}$. \square

Proof of Theorem 5.6. Let $\Theta \in \mathcal{O}_3(T^*[2]M \oplus E[1])$ be the Hamiltonian generating function of $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$. By Proposition 5.4, a Dirac generating operator $D \in \mathcal{D}_3^-(M, \mathbb{S})_{\mathbb{R}}$ of $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ satisfies $\sigma_3(D) = \Theta$. According to Lemma 5.7, there exists a unique such D satisfying in addition $D^* = -D$, and it is given by $D = \mathcal{WQ}(\Theta)$. \square

Next we will describe an explicit formula for the skew-symmetric Dirac generating operator D . By uniqueness, D does not depend on any choice of connections. However, the Weyl quantization map \mathcal{WQ} does. This is reflected in the formula (5.7) below giving D , which is written in terms of connections $\nabla^{\mathbb{S}}$ on \mathbb{S} and ∇ on $E \otimes TM$, both of which are induced by a connection on TM and compatible connections on S and E . In addition, the formula (5.7) involves the torsion $C_{\nabla} \in \Gamma(\wedge^3 E)$ (see Eq. (5.4)), a basis of local sections (ξ^a) of E and the metric components g_{ab} , defined by $g^{bc} = g(\xi^b, \xi^c)$ and $g_{ab}g^{bc} = \delta_a^c$.

Corollary 5.8. *Let $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ be a Courant algebroid. The skew-symmetric Dirac generating operator is given by the following formula*

$$(5.7) \quad D = \frac{1}{\sqrt{2}} \left[g_{ab} \gamma(\xi^a) \nabla_{\rho(\xi^b)}^{\mathbb{S}} + \frac{1}{2} \gamma(C_{\nabla}) + \frac{1}{2} \gamma(\text{Tr } \nabla \rho) \right],$$

where $\nabla \rho \in \Gamma(E \otimes TM \otimes T^*M)$ indeed admits a trace (see (4.13)).

Proof. By Theorem 5.6 and Theorem 5.2, we know that $D = \mathcal{WQ}(\Theta)$ with $\Theta = \rho + C_{\nabla}$. The result follows from Proposition 4.10. \square

5.4. An alternative formula for Dirac generating operators. We now prove that the Dirac generating operator constructed by Alekseev and Xu in [1] coincides with the operator given by Eq. (5.7). We need to introduce several notations, directly borrowed from [1].

Definition 5.9. [1] *Let $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ be a Courant algebroid and $V \rightarrow M$ a vector bundle. A \mathbb{R} -bilinear map $\nabla : \Gamma(E) \otimes_{\mathbb{R}} \Gamma(V) \rightarrow \Gamma(V)$ is called an E -connection on V if it satisfies the following properties,*

$$\begin{aligned} \nabla_{f\xi} v &= f \nabla_{\xi} v, \\ \nabla_{\xi}(fv) &= f \nabla_{\xi} v + \rho(\xi)[f] \cdot v, \end{aligned}$$

for all $\xi \in \Gamma(E)$, $v \in \Gamma(V)$ and $f \in C^{\infty}(M)$.

Remark 5.10. An ordinary connection ∇ on V induces an E -connection in the following way

$$(5.8) \quad \nabla_{\xi} v := \nabla_{\rho(\xi)} v, \quad \forall \xi \in \Gamma(E), v \in \Gamma(V).$$

Let ∇^E be an E -connection on E . According to [1],

$$(5.9) \quad C_{\nabla}(\xi_1, \xi_2, \xi_3) = \frac{1}{2} \text{cycl}_{123} \, g \left(\frac{1}{3} (\llbracket \xi_1, \xi_2 \rrbracket - \llbracket \xi_2, \xi_1 \rrbracket) - (\nabla_{\xi_1}^E \xi_2 - \nabla_{\xi_2}^E \xi_1), \xi_3 \right)$$

defines a section of $\Gamma(\wedge^3 E)$, where cycl_{123} denotes the sum over cyclic permutations. This is called the torsion of E with respect to ∇^E . If ∇^E is induced from an ordinary connection ∇^E as in Eq. (5.8), then C_{∇} coincides with the torsion C_{∇} , introduced previously in Eq. (5.4).

Let (ξ^a) be a basis of local sections of E . For any $\xi \in \Gamma(E)$, the function $\text{Div } \xi \in C^\infty(M)$, defined by

$$(5.10) \quad \text{Div } \xi = \sum_a g(\nabla_{\xi^a}^E \xi, \xi^a),$$

does not depend on the chosen local basis (ξ^a) . We call $\text{Div } \xi$ the divergence of ξ . Then,

$$(5.11) \quad \nabla_{\xi}^{\Lambda} s := L_{\rho(\xi)} s - (\text{Div } \xi) s$$

defines an E -connection on $\wedge^{\text{top}} T^*M$. Here $L_{\rho(\xi)} s$ stands for the Lie derivative of $s \in \Gamma(\wedge^{\text{top}} T^*M)$ along the vector field $\rho(\xi) \in \mathfrak{X}(M)$.

Lemma 5.11. *Let ∇^E be a metric connection on E . It induces an E -connection on E as in Eq. (5.8) and an E -connection ∇^{Λ} on $\wedge^{\text{top}} T^*M$ via Eq. (5.11). For any affine connection ∇^{TM} on TM , we have the relation*

$$\nabla_{\xi}^{\Lambda} = \nabla_{\rho(\xi)}^{\Lambda} + g(\xi, \text{Tr } \nabla \rho), \quad \forall \xi \in \Gamma(E),$$

where ∇^{Λ} and ∇ are induced connections on $\wedge^{\text{top}} T^*M$ and $E \otimes TM$ respectively.

Proof. The Lie derivative on $\wedge^{\text{top}} T^*M$ satisfies the equation :

$$(5.12) \quad L_{\rho(\xi)} = \nabla_{\rho(\xi)}^{\Lambda} + \text{Tr } \nabla^{TM}(\rho(\xi)).$$

Since ∇^E preserves the metric, we have $\nabla^{TM}(\rho(\xi)) = g(\nabla \rho, \xi) + g(\rho, \nabla^E \xi)$. This is an equality in $\Gamma(TM \otimes T^*M)$, so that we can take the trace,

$$(5.13) \quad \text{Tr } \nabla^{TM}(\rho(\xi)) = g(\text{Tr } \nabla \rho, \xi) + \text{Div } \xi,$$

where the divergence of ξ is computed for the E -connection on E induced by ∇^E . The conclusion follows by combination of Eqns (5.11), (5.12) and (5.13). \square

An E -connection on E preserves the metric if

$$\rho(\xi_1)[g(\xi_2, \xi_3)] = g(\nabla_{\xi_1}^E \xi_2, \xi_3) + g(\xi_2, \nabla_{\xi_1}^E \xi_3),$$

for all $\xi_1, \xi_2, \xi_3 \in \Gamma(E)$. An E -connection ∇^E on E and an E -connection ∇^S on S are compatible if ∇^E preserves the metric and both E -connections induce the same E -connection on the Clifford algebra bundle $\text{Cl}(E) \cong \text{End } S$. By Eq. (5.11), such a pair of compatible E -connections induces an E -connection on the twisted spinor bundle $\mathbb{S} = S \otimes (\det S^*)^{1/N} \otimes |\wedge^{\text{top}} T^*M|^{1/2}$.

Proposition 5.12. *Let $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ be a Courant algebroid with twisted spinor bundle \mathbb{S} . The unique skew-symmetric Dirac generating operator is given by*

$$(5.14) \quad D = \frac{1}{\sqrt{2}} \left[g_{ab} \gamma(\xi^a) \nabla_{\xi^b}^{\mathbb{S}} + \frac{1}{2} \gamma(C_{\nabla}) \right],$$

where $\nabla^{\mathbb{S}}$ is the E -connection on \mathbb{S} induced by compatible E -connections on S and E . Here, (ξ^a) is a basis of local sections of E and g_{ab} are the metric components.

Proof. According to [1, Theorem 4.3], the formula (5.14) does not depend on the choice of E -connections on E and S . Hence, we can start with E -connections induced by ordinary connections. In this case, we have $C_{\nabla} = C_{\nabla}$ and $\nabla_{\xi}^{\mathbb{S}} = \nabla_{\rho(\xi)}^{\mathbb{S}} + g(\xi, \text{Tr } \nabla \rho)$ by Lemma 5.11. Substituting both equalities in Eq. (5.14) yields Eq. (5.7). The result follows. \square

5.5. A Courant algebroids invariant. Since a skew-symmetric generating operator D is unique, then $D^2 \in C^\infty(M)$ is an invariant of the Courant algebroid. It is natural to ask how this invariant can be described geometrically.

Let $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ be a Courant algebroid. By abuse of notation, we denote by g the metric on $\wedge E$ induced by the metric on E as follows, $g(\xi, \eta) = \det(g(\xi_i, \eta_j))$ for all $\xi = \xi_1 \wedge \cdots \wedge \xi_k$ and $\eta = \eta_1 \wedge \cdots \wedge \eta_k$ in $\Gamma(\wedge^k E)$. Using a metric connection ∇^E on E and an affine connection on TM , we define $f_E \in C^\infty(M)$ by

$$(5.15) \quad f_E = g(C_{\nabla}, C_{\nabla}) - g(\text{Tr } \nabla \rho, \text{Tr } \nabla \rho) - 2 \text{Div}(\text{Tr } \nabla \rho),$$

where C_{∇} is the torsion defined by Eq. (5.4), ∇ is the induced connection on $E \otimes TM$ and $\text{Div}(\text{Tr } \nabla \rho) = \sum_a g(\xi^a, \nabla_{\rho(\xi^a)}^E (\text{Tr } \nabla \rho))$ is the divergence of $\text{Tr } \nabla \rho \in \Gamma(E)$, see Eq. (5.10).

Theorem 5.13. *The function f_E defined by Eq. (5.15) satisfies the following properties:*

- (1) f_E is independent of the choice of the connections ∇ and ∇^E ;
- (2) $\{\Theta, f_E\} = 0$, where $\Theta = \rho + C_{\nabla}$ is the Hamiltonian generating function of $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$;
- (3) if (E, g) admits a twisted spinor bundle, then $f_E = -8D^2$, where D is the skew-symmetric Dirac generating operator of $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$.

Proof. In view of definition (5.15) of the function f_E , it is sufficient to prove the three above properties locally, over a contractible open subset $U \subset M$.

Restricted to U , a pseudo-Euclidean vector bundle $(E|_U, g)$ always admits a twisted spinor bundle $\mathbb{S}|_U$. According to Theorem 5.6, we obtain a skew-symmetric Dirac generating operator $D \in \mathcal{D}(U, \mathbb{S}|_U)_{\mathbb{R}}$. By a straightforward computation using Eq. (5.7), we check that $f_E = -8D^2$. The other claims follow, in particular $\{\Theta, f_E\} = -8\sigma_2([D, D^2]) = 0$. \square

As a consequence, the function f_E is an invariant of the Courant algebroid. Our next goal is to provide an intrinsic formula for this function. We first recall some standard facts on the structure of a Courant algebroid $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$. It is well-known that $(\ker \rho)^\perp \subset \ker \rho$ (see e.g. [30]). Moreover, $(\ker \rho)^\perp$ and $\ker \rho$ are two-sided ideals in E for the Dorfman bracket. Hence, the quotient

$$\mathcal{G} := \ker \rho / (\ker \rho)^\perp$$

is a bundle of quadratic Lie algebras, whose fiberwise Lie bracket $[\cdot, \cdot]^{\mathcal{G}}$ and non-degenerate ad-invariant scalar product $(\cdot, \cdot)^{\mathcal{G}}$ are inherited from the Dorfman bracket $\llbracket \cdot, \cdot \rrbracket$ and the metric g (for further details, see [9]). Since fibers of \mathcal{G} are quadratic Lie algebras, we have an analogue of Cartan 3-form $C_{\mathcal{G}} \in \Gamma(\wedge^3 \mathcal{G})$:

$$C_{\mathcal{G}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) := ([\mathbf{r}_1, \mathbf{r}_2]^{\mathcal{G}}, \mathbf{r}_3)^{\mathcal{G}},$$

for all $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \Gamma(\mathcal{G})$. If ρ is of constant rank over a neighborhood of a point $x \in M$, we say that the Courant algebroid is regular at the point x . In this case, $C_{\mathcal{G}}$ is smooth in a neighborhood of x . Note that regular points form a dense open subset M_{reg} of the base manifold M .

Theorem 5.14. *Let $(E, g, \rho, \llbracket \cdot, \cdot \rrbracket)$ be a Courant algebroid. On M_{reg} , we have*

$$f_E = g(C_{\mathcal{G}}, C_{\mathcal{G}}).$$

Proof. As both sides of the above equality depend only on the local structure of the Courant algebroid, we can assume that E is a regular Courant algebroid.

We recall some useful constructions regarding regular Courant algebroids given in [9]. The vector bundle E admits a splitting:

$$(5.16) \quad E \cong F^* \oplus \mathcal{G} \oplus F,$$

where $F = \rho(E) \subset TM$ is an integrable distribution. Under the above isomorphism, the anchor map ρ becomes projection $\rho : E \rightarrow F$ and each section $\xi \in \Gamma(E)$ decomposes as $\xi = \eta + \mathbf{r} + x$, with $\eta \in \Gamma(F^*)$, $\mathbf{r} \in \Gamma(\mathcal{G})$ and $x \in \Gamma(F)$. In the reminder of the proof, we use the decomposition $\xi_i = \eta_i + \mathbf{r}_i + x_i$, for all $\xi_i \in \Gamma(E)$. Let ∇^{TM} be a torsion-free affine connection on TM and denote by ∇^F and ∇^{F^*} the induced connections on F and F^* respectively. According to [9, Proposition 1.1] and [9, Lemma 1.4], there exists a metric F -connection $\nabla^{\mathcal{G}} : \Gamma(F) \otimes \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G})$ and a section $\mathcal{H} \in \Gamma(\wedge^3 F^*)$ such that the formula

$$(5.17) \quad \nabla'_{\xi_1} \xi_2 = (\nabla_{x_1}^{F^*} \eta_2 - \frac{1}{3} \mathcal{H}(x_1, x_2, \cdot)) + (\nabla_{x_1}^{\mathcal{G}} \mathbf{r}_2 + \frac{2}{3} [\mathbf{r}_1, \mathbf{r}_2]^{\mathcal{G}}) + \nabla_{x_1}^F x_2, \quad \forall \xi_1, \xi_2 \in \Gamma(E),$$

defines a E -connection ∇' on E . As shown in [9, Formula (3.1)], the torsion of ∇' reads as

$$(5.18) \quad C_{\nabla'}(\xi_1, \xi_2, \xi_3) = \mathcal{H}(x_1, x_2, x_3) + \text{cycl}_{123} (R(x_1, x_2), \mathbf{r}_3)^{\mathcal{G}} - C_{\mathcal{G}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3),$$

for all $\xi_i \in \Gamma(E)$, $i = 1, 2, 3$, where $R : \wedge^2 F \rightarrow \mathcal{G}$ is a certain bundle map.

To compute f_E via Eq. (5.15), we need a metric connection on E and an affine connection on TM . For the latter, we choose the above connection ∇^{TM} . As for the metric connection on E , we define it as follows. Pick a Riemannian metric on M and denote by $F^{\perp} \subset TM$ the orthogonal distribution to F . Choose a metric connection $\tilde{\nabla}^{\mathcal{G}}$ on \mathcal{G} and define a new metric connection on \mathcal{G} by setting

$$\nabla_X^{\mathcal{G}} \mathbf{r} := \nabla_{X_F}^{\mathcal{G}} (\mathbf{r}) + \tilde{\nabla}_{X_{F^{\perp}}}^{\mathcal{G}} (\mathbf{r}), \quad \forall X \in \mathfrak{X}(M), \mathbf{r} \in \Gamma(\mathcal{G}),$$

where the splitting $X = X_F + X_{F^\perp}$ corresponds to the decomposition $TM = F \oplus F^\perp$. A metric connection on E can then be defined by the following formula

$$(5.19) \quad \nabla^E := \nabla^{F^*} \oplus \nabla^{\mathcal{G}} \oplus \nabla^F.$$

Since the anchor map ρ is identified with the projection onto F through the decomposition (5.16), it is simple to check that

$$\nabla \rho = \nabla^{TM} \circ \rho - \rho \circ \nabla^E = 0.$$

According to Eq. (5.15), then we have

$$(5.20) \quad f_E = g(C_\nabla, C_\nabla),$$

with C_∇ being the torsion of E with respect to ∇^E .

Our next goal is to compute C_∇ . The connection ∇^E defined in Eq. (5.19) induces an E -connection given explicitly by

$$\nabla_{\xi_1} \xi_2 = \nabla_{x_1}^{F^*} \eta_2 + \nabla_{x_1}^{\mathcal{G}} \mathbf{r}_2 + \nabla_{x_1}^F x_2, \quad \forall \xi_1, \xi_2 \in \Gamma(E).$$

In view of Eq. (5.17), the E -connections ∇ and ∇' are related as follows:

$$\nabla'_{\xi_1} \xi_2 - \nabla_{\xi_1} \xi_2 = -\frac{1}{3} \mathcal{H}(x_1, x_2, \cdot) + \frac{2}{3} [\mathbf{r}_1, \mathbf{r}_2]^{\mathcal{G}}.$$

Using the fact that $C_\nabla = C_{\nabla'}$ and Eq. (5.9), we deduce that

$$\begin{aligned} C_\nabla(\xi_1, \xi_2, \xi_3) - C_{\nabla'}(\xi_1, \xi_2, \xi_3) &= \text{cycl}_{123} \, g \left(-\frac{1}{3} \mathcal{H}(x_1, x_2, \cdot) + \frac{2}{3} [\mathbf{r}_1, \mathbf{r}_2]^{\mathcal{G}}, \xi_3 \right), \\ &= -\frac{1}{3} \text{cycl}_{123} (\mathcal{H}(x_1, x_2, x_3)) + \frac{2}{3} \text{cycl}_{123} \, g([\mathbf{r}_1, \mathbf{r}_2]^{\mathcal{G}}, \mathbf{r}_3), \\ &= -\mathcal{H}(x_1, x_2, x_3) + 2C_{\mathcal{G}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \end{aligned}$$

for all $\xi_i \in \Gamma(E)$, $i = 1, 2, 3$. Therefore, by Eq. (5.18), we have

$$C_\nabla(\xi_1, \xi_2, \xi_3) = C_{\mathcal{G}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \text{cycl}_{123} (R(x_1, x_2), \mathbf{r}_3)^{\mathcal{G}}.$$

This means that $C_\nabla - C_{\mathcal{G}} \in \Gamma(\wedge^2 F \otimes \mathcal{G})$. Using the fact that $g(F, F \oplus \mathcal{G}) = 0$ and Eq. (5.20), we obtain $f_E = g(C_{\mathcal{G}}, C_{\mathcal{G}})$. \square

Remark 5.15. By definition, the function f_E is always smooth, even for a non-regular Courant algebroid E . Therefore, one can consider the function f_E as the smooth extension of $g(C_{\mathcal{G}}, C_{\mathcal{G}})$ from M_{reg} to M . There is no reason to expect that the function $g(C_{\mathcal{G}}, C_{\mathcal{G}})$ itself is smooth at the non-regular points of E , but we were unable to find an example of Courant algebroid such that $g(C_{\mathcal{G}}, C_{\mathcal{G}}) \neq f_E$.

6. APPLICATIONS TO LIE BIALGEBROIDS

6.1. Weyl quantization in the splittable case. From now on, we assume that the pseudo-Euclidean vector bundle (E, g) splits as $E := A \oplus A^*$ and the pseudo-metric is given by

$$(6.1) \quad g(\zeta_1 + \eta_1, \zeta_2 + \eta_2) = \frac{1}{2} \langle \zeta_1, \eta_2 \rangle + \frac{1}{2} \langle \zeta_2, \eta_1 \rangle, \quad \forall \zeta_1, \zeta_2 \in \Gamma(A), \eta_1, \eta_2 \in \Gamma(A^*),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between A and A^* . The previous constructions can then be described more explicitly.

According to Example 3.1, $S_{\mathbb{R}} := \wedge A^*$ is a real spinor bundle of (E, g) . Under the identification

$$\Gamma(S_{\mathbb{R}}) = \mathcal{O}(A[1]),$$

the Clifford action of $\Gamma(E)$, given in (3.3), reads as

$$(6.2) \quad \gamma(\zeta)\phi = \langle \zeta, \eta^a \rangle \left(\frac{\partial}{\partial \eta^a} \phi \right) \quad \text{and} \quad \gamma(\eta)\phi = \eta\phi, \quad \forall \zeta \in \Gamma(A), \eta \in \Gamma(A^*), \phi \in \mathcal{O}(A[1]),$$

where (η^a) is a basis of local sections of A^* . We will need the following relations later on,

$$(6.3) \quad \gamma(\psi)\phi = \psi\phi \quad \text{and} \quad \gamma(\psi\zeta)\phi = \left(\langle \zeta, \eta^a \rangle \psi \frac{\partial}{\partial \eta^a} + \frac{(-1)^\kappa}{2} \langle \zeta, \psi \rangle \right) \phi,$$

for all $\zeta \in \Gamma(A)$, $\psi \in \mathcal{O}_\kappa(A[1])$, $\phi \in \mathcal{O}(A[1])$. Assume that the line bundle $\mathcal{L} := \wedge^{\text{top}} A \otimes \wedge^{\text{top}} T^*M$ admits a square root and set $\mathbb{S}_{\mathbb{R}} := \wedge A^* \otimes \mathcal{L}^{1/2}$. Since $\det S_{\mathbb{R}} \cong (\wedge^{\text{top}} A^*)^N$, with N being the rank of $\wedge A^*$, the \mathbb{C} -vector bundle $\mathbb{S} := \mathbb{S}_{\mathbb{R}} \otimes \mathbb{C}$ defines a twisted spinor bundle (see Section 3.3). Accordingly, the \mathbb{R} -vector bundle $\mathbb{S}_{\mathbb{R}}$ is called a *real twisted spinor bundle*. By pull-back along $\pi : A[1] \rightarrow M$, the line bundle $\pi^*\mathcal{L}$ can be identified with the Berezinian line bundle $\text{Ber}_A \rightarrow A[1]$ (see e.g. [19, 14]). Therefore, we obtain an isomorphism of $\mathcal{O}(A[1])$ -modules

$$(6.4) \quad v : \Gamma(\mathbb{S}_{\mathbb{R}}) \xrightarrow{\sim} \Gamma(\text{Ber}_A^{1/2}).$$

A connection on A induces a metric connection ∇^E on E and a compatible spinor connection ∇^S on $S_{\mathbb{R}}$. In addition, a connection on TM induces a spinor connection $\nabla^{\mathbb{S}}$ on $\mathbb{S}_{\mathbb{R}}$, compatible with ∇^E , and a connection ∇ on the line bundle $\text{Ber}_A^{1/2} \rightarrow A[1]$. The latter can be defined as follows. First, note that the space of vector fields $\mathfrak{X}(A[1])$ is linearly spanned by the vector fields X which, as derivations of $\mathcal{O}(A[1]) = \Gamma(S_{\mathbb{R}})$, take the form

$$(6.5) \quad X = f \cdot \nabla_{X_0}^S + g \cdot \gamma(\zeta),$$

with $f, g \in \mathcal{O}(A[1])$, $X_0 \in \mathfrak{X}(M)$ and $\zeta \in \Gamma(A)$. By setting

$$\nabla_X := v \circ \left(f \cdot \nabla_{X_0}^{\mathbb{S}} + g \cdot \gamma(\zeta) \right) \circ v^{-1},$$

one obtains a well-defined connection ∇ on $\text{Ber}_A^{1/2}$.

The algebra $\mathcal{D}(A[1], \text{Ber}_A^{1/2})$, of differential operators on $\text{Ber}_A^{1/2}$, is a subalgebra of $\text{End}(\Gamma(\text{Ber}_A^{1/2}))$ generated by multiplication operators by functions on $A[1]$ and covariant derivatives ∇_X , $\forall X \in \mathfrak{X}(A[1])$.

Lemma 6.1. *The map defined by $\Upsilon_A(D) := v \circ D \circ v^{-1}$, $\forall D \in \mathcal{D}(M, \mathbb{S}_{\mathbb{R}})$, provides an isomorphism of algebras*

$$\Upsilon_A : \mathcal{D}(M, \mathbb{S}_{\mathbb{R}}) \xrightarrow{\sim} \mathcal{D}(A[1], \text{Ber}_A^{1/2}).$$

Proof. By Eq. (6.4), the map $\Upsilon_A(\cdot) = v \circ \cdot \circ v^{-1}$ is an algebra isomorphism between $\text{End}(\Gamma(\mathbb{S}_{\mathbb{R}}))$ and $\text{End}(\Gamma(\text{Ber}_A^{1/2}))$. Since generators of $\mathcal{D}(M, \mathbb{S}_{\mathbb{R}})$ and $\mathcal{D}(A[1], \text{Ber}_A^{1/2})$ are clearly in bijection via Υ_A , the result follows. \square

Since the metric g in Eq. (6.1) is half the duality pairing, by Lemma 2.3, we have a symplectic diffeomorphism $\tilde{\Xi}_{\nabla}$ between $T^*[2](A[1])$ and $(T^*[2]M \oplus E[1], \omega_{2g, \nabla E})$. Therefore, the map $\Phi_A := (\tilde{\Xi}_{\nabla})^{-1} \circ (\text{id}_{T^*M} \oplus \sqrt{2} \text{id}_E)$ defines a symplectic diffeomorphism:

$$\Phi_A : T^*[2]M \oplus E[1] \xrightarrow{\sim} T^*[2]A[1].$$

Consider the Weyl quantization map $\mathcal{WQ} : \mathcal{O}(T^*[2]M \oplus E[1]) \rightarrow \mathcal{D}(M, \mathbb{S}_{\mathbb{R}})$ (see Proposition 4.12). According to Remark 3.19, the restriction from \mathbb{S} to $\mathbb{S}_{\mathbb{R}}$ induces an algebra isomorphism

$$\mathcal{R} : \mathcal{D}(M, \mathbb{S}_{\mathbb{R}}) \xrightarrow{\sim} \mathcal{D}(M, \mathbb{S}).$$

Definition 6.2. *The Weyl quantization on $T^*[2](A[1])$ is the map*

$$\mathcal{WQ}^A : \mathcal{O}(T^*[2](A[1])) \longrightarrow \mathcal{D}(A[1], \text{Ber}_A^{1/2}),$$

which is given by $\mathcal{WQ}^A := \Upsilon_A \circ \mathcal{R} \circ \mathcal{WQ} \circ (\Phi_A)^$.*

Note that \mathcal{WQ}^A is a linear isomorphism.

Remark 6.3. Connections on A and TM give rise to a connection ∇ on $T(A[1]) \rightarrow A[1]$. It would be interesting to compare \mathcal{WQ}^A with the Weyl quantization constructed directly from the induced connections on $T(A[1]) \rightarrow A[1]$ and $\text{Ber}_A^{1/2} \rightarrow A[1]$ via Eq. (4.7).

There is a notion of Lie derivative on the space of sections $\Gamma(\text{Ber}_A^{1/2})$ (see e.g. [19]). We will need its expression in local affine coordinates (x^i, η^a) on $A[1]$. Assume that $X \in \mathfrak{X}(A[1])$ is a vector field of degree κ . Using the local expression $X = X^i \frac{\partial}{\partial x^i} + X^a \frac{\partial}{\partial \eta^a}$, where X^i and X^a are local functions on $A[1]$, the Lie derivative on $\text{Ber}_A^{1/2}$ reads as

$$(6.6) \quad L_X = X^i \frac{\partial}{\partial x^i} + X^a \frac{\partial}{\partial \eta^a} + \frac{1}{2} \left(\frac{\partial}{\partial x^i} X^i + (-1)^{\kappa+1} \frac{\partial}{\partial \eta^a} X^a \right),$$

in the trivialization of $\text{Ber}_A^{1/2}$ provided by $(\wedge dx^i \otimes \prod \zeta_a)^{1/2}$, with (ζ_a) the dual frame of (η^a) .

Proposition 6.4. *For any vector field $X \in \mathfrak{X}(A[1])$, we have*

$$(6.7) \quad \mathcal{WQ}^A(F_X) = L_X,$$

where F_X is the fiberwise linear function on $T^[2](A[1])$ corresponding to X .*

Proof. Let $X \in \mathfrak{X}(A[1])$ be a vector field of degree κ . It suffices to prove Eq. (6.7) at each point $x \in M$. Consider local affine coordinates (x^i, η^a) on $A[1]$ obtained by pull-back of a Cartesian coordinate system on $T_x M \times A_x$, via the map \mathcal{T}_x defined in Eq. (4.3). This system of coordinates induce fiberwise coordinates (ζ_a) of A and (p_i) of T^*M . Then, $(x^i, \eta^a, p_i, \zeta_a)$ is a local coordinate system of both $T^*[2](A[1])$ and $T^*[2]M \oplus E[1]$. By Eq. (2.5), the map $\Phi_A : T^*[2]M \oplus E[1] \rightarrow T^*[2]A[1]$ satisfies then

$$(\Phi_A)^* x^i = x^i, \quad (\Phi_A)^* p_i^{\nabla} = p_i, \quad (\Phi_A)^* \eta^a = \sqrt{2} \eta^a, \quad (\Phi_A)^* \zeta_a = \sqrt{2} \zeta_a,$$

where p_i^∇ is the fiberwise linear function on $T^*[2](A[1])$ corresponding to the vector field ∇_i^A on $A[1]$. Note that $\nabla_i^A = \frac{\partial}{\partial x^i} + \Gamma_{ib}^a \eta^b \frac{\partial}{\partial \eta^a}$, where the real function Γ_{ib}^a vanishes at x by definition of the coordinates (x^i, η^a) . Using the local expression $X = X^i \frac{\partial}{\partial x^i} + X^a \frac{\partial}{\partial \eta^a}$, we obtain

$$(\Phi_A)^* F_X = (\sqrt{2})^\kappa \left(X^i \cdot (p_i + \Gamma_{ib}^a \eta^b \zeta_a) + 2X^a \zeta_a \right).$$

From Eqns (4.15)-(4.17) and $\Gamma_{ib}^a(x) = 0$, we deduce that

$$\left(\mathcal{WQ}^A(F_X) \psi \right) (x, \eta) = \left(\gamma \left(X^i(x, \eta) \right) \frac{\partial}{\partial x^i} + \frac{1}{2} \gamma \left(\frac{\partial}{\partial x^i} X^i(x, \eta) \right) + \gamma \left(X^a(x, \eta) \zeta_a \right) \right) \psi(x, \eta),$$

in the trivialization of $\text{Ber}_A^{1/2}$ provided by $(\wedge dx^i \otimes \prod \zeta_a)^{1/2}$. The result follows then from Eqns (6.3) and (6.6). \square

Remark 6.5. Let $X \in \mathfrak{X}(A[1])$ be a vector field as in Eq. (6.5). According to Eqns (4.18) and (6.7), the Lie derivative on $\text{Ber}_A^{1/2}$ is given by

$$L_X = \nabla_X + \frac{1}{2} \left(\text{Tr } \nabla(fX_0) + \langle \zeta, g \rangle \right).$$

Remark 6.6. Let $\alpha' : \mathcal{O}^{\mathbb{C}}(A[1]) \rightarrow \mathcal{O}^{\mathbb{C}}(A[1])$ be the $C^\infty(M)$ -linear antiautomorphism satisfying $\alpha'(\phi_0) = \phi_0$ if $\phi_0 \in \mathcal{O}_1^{\mathbb{C}}(A[1])$. The map α' extends naturally to $\Gamma(\text{Ber}_A^{1/2}) \otimes \mathbb{C} \cong \mathcal{O}^{\mathbb{C}}(A[1]) \otimes_{C^\infty} \Gamma(\mathcal{L}^{1/2})$ by setting $\alpha := \alpha' \otimes \text{id}$. Let

$$\varepsilon = \begin{cases} 1 & \text{if } \text{rk } A \equiv 0, 1 \pmod{4}, \\ i & \text{if } \text{rk } A \equiv 2, 3 \pmod{4}. \end{cases}$$

Via the Berezin integration over the supermanifold $A[1]$, we define a pseudo-Hermitian scalar product

$$(\phi, \psi) = \varepsilon \int_{A[1]} \overline{\alpha(\phi)} \psi,$$

on the space of compactly supported sections $\Gamma_c(\text{Ber}_A^{1/2}) \otimes \mathbb{C}$. Under the isomorphism $v \otimes \text{id}_{\mathbb{C}} : \Gamma(\mathbb{S}) \xrightarrow{\sim} \Gamma(\text{Ber}_A^{1/2}) \otimes \mathbb{C}$, the above scalar product on $\text{Ber}_A^{1/2} \otimes \mathbb{C}$ furnishes a globalization of the local spinor scalar product on \mathbb{S} given in Eq. (3.22). Accordingly, the adjoint operation on $\mathcal{D}(M, \mathbb{S})$, defined in Proposition 3.16, coincides with the one defined on $\mathcal{D}(A[1], \text{Ber}_A^{1/2} \otimes \mathbb{C})$ by the above scalar product.

6.2. Preliminaries on Lie algebroids. Now let A be a Lie algebroid. According to [11], the line bundle $\mathcal{L} = \wedge^{\text{top}} A \otimes \wedge^{\text{top}} T^*M$ carries a representation of the Lie algebroid $(A, [\cdot, \cdot], \rho)$, given by

$$D_\zeta(v \otimes s) := [\zeta, v] \otimes s + v \otimes L_{\rho(\zeta)} s,$$

with $\zeta \in \Gamma(A)$, $v \in \Gamma(\wedge^{\text{top}} A)$ and $s \in \Gamma(\wedge^{\text{top}} T^*M)$. Here the bracket stands for the Schouten bracket on $\Gamma(\wedge A)$, and $L_{\rho(\zeta)} s$ stands for the Lie derivative of $s \in \Gamma(\wedge^{\text{top}} T^*M)$ along the vector field $\rho(\zeta) \in \mathfrak{X}(M)$. Since the map $\zeta \mapsto D_\zeta$ is $C^\infty(M)$ -linear, for each section $v \otimes s \in \Gamma(\mathcal{L})$, there exists a unique $\eta_0 \in \Gamma(A^*)$ such that

$$(6.8) \quad D_\zeta(v \otimes s) = \langle \eta_0, \zeta \rangle v \otimes s, \quad \forall \zeta \in \Gamma(A).$$

The element η_0 is called the modular cocycle of the Lie algebroid $(A, [\cdot, \cdot], \rho)$ associated to the section $v \otimes s \in \Gamma(\mathcal{L})$. Assume the square root of \mathcal{L} exists. Then, $\mathcal{L}^{1/2}$ admits an action of the Lie algebroid A , defined by

$$(6.9) \quad \tilde{D}_\zeta((v \otimes s)^{1/2}) := \frac{1}{2} \langle \eta_0, \zeta \rangle (v \otimes s)^{1/2}.$$

The Chevalley-Eilenberg differential Q of the Lie algebroid A is a differential of the complex $\bigoplus_k \Gamma(\wedge^k A^*)$, namely

$$\begin{aligned} Q(\phi_0)(\zeta_0, \dots, \zeta_k) &= \sum_{a=0}^{\text{rk } A} (-1)^a \rho(\zeta_a) \left[\phi_0(\zeta_0, \dots, \hat{\zeta}_a, \dots, \zeta_k) \right] \\ &\quad + \sum_{a < b} (-1)^{a+b} \phi_0([\zeta_a, \zeta_b], \zeta_0, \dots, \hat{\zeta}_a, \dots, \hat{\zeta}_b, \dots, \zeta_k), \end{aligned}$$

where $\phi_0 \in \Gamma(\wedge^k A^*)$ and $\zeta_0, \dots, \zeta_k \in \Gamma(A)$. Equivalently, Q is a homological vector field on $A[1]$. In local coordinates (x^i, η^a) , the expression of Q is

$$(6.10) \quad Q = \rho_a^i \eta^a \frac{\partial}{\partial x^i} + \frac{1}{2} C_{bc}^a \eta^c \eta^b \frac{\partial}{\partial \eta^a},$$

where, by definition,

$$\rho_a^i = \langle \rho(\zeta_a), dx^i \rangle \quad \text{and} \quad C_{bc}^a = \langle [\zeta_b, \zeta_c], \eta^a \rangle$$

are smooth functions on the base manifold and (ζ_a) is the dual frame to (η^a) . From Eq. (6.9), we obtain another differential \tilde{Q} , defined on the complex $\bigoplus_k \Gamma(\wedge^k A^* \otimes \mathcal{L}^{1/2}) \cong \Gamma(\text{Ber}_A^{1/2})$:

$$\begin{aligned} \tilde{Q}(\phi)(\zeta_0, \dots, \zeta_k) &= \sum_{a=0}^{\text{rk } A} (-1)^a \tilde{D}_{\zeta_a} \left(\phi(\zeta_0, \dots, \hat{\zeta}_a, \dots, \zeta_k) \right) \\ &\quad + \sum_{a < b} (-1)^{a+b} \phi([\zeta_a, \zeta_b], \zeta_0, \dots, \hat{\zeta}_a, \dots, \hat{\zeta}_b, \dots, \zeta_k), \end{aligned}$$

where $\phi \in \Gamma(\wedge^k A^* \otimes \mathcal{L}^{1/2})$ and $\zeta_0, \dots, \zeta_k \in \Gamma(A)$.

Proposition 6.7. *Let $Q \in \mathfrak{X}(A[1])$ be the homological vector field of the Lie algebroid A . As an operator on $\Gamma(\text{Ber}_A^{1/2})$, the differential \tilde{Q} satisfies*

$$(6.11) \quad \tilde{Q} = L_Q.$$

If $\phi = \phi_0 \otimes (v \otimes s)^{1/2} \in \Gamma(\wedge A^* \otimes \mathcal{L}^{1/2})$, we have

$$(6.12) \quad L_Q(\phi) = (Q(\phi_0) + \frac{1}{2} \eta_0 \phi_0) \otimes (v \otimes s)^{1/2},$$

where $\eta_0 \in \Gamma(A^*)$ is the modular cocycle of the Lie algebroid A associated to the section $v \otimes s$.

Proof. Let $\phi = \phi_0 \otimes (v \otimes s)^{1/2} \in \Gamma(\wedge A^* \otimes \mathcal{L}^{1/2})$. A direct computation shows that $\tilde{Q}\phi = (Q + \frac{1}{2} \eta_0) \phi_0 \otimes (v \otimes s)^{1/2}$. It suffices then to prove that $L_Q = \tilde{Q}$ in a local coordinate system (x^i, η^a) on $A[1]$. We work in the trivialization of $\mathcal{L}^{1/2}$ provided by the local section $(\wedge dx^i \otimes \prod \zeta_a)^{1/2}$, with (ζ_a) the dual frame to (η^a) . By Eqns (6.2), (6.6) and (6.10), we have

$$L_Q = Q + \frac{1}{2} \left(\frac{\partial}{\partial x^i} \rho_b^i + C_{ba}^a \right) \eta^b.$$

By Eq. (6.8), the modular cocycle w.r.t. the local section $(\wedge dx^i \otimes \prod \zeta_a)$ satisfies $\eta_0 = \frac{\partial}{\partial x^i} \rho_b^i + C_{ba}^a$. The result follows. \square

Remark 6.8. Each section $v \otimes s \in \Gamma(\mathcal{L})$ defines a section of the Berezinian bundle $1 \otimes (v \otimes s) \in \Gamma(\text{Ber}_A)$ and then a divergence by the formula: $\text{Div } X = \frac{L_X(1 \otimes v \otimes s)}{1 \otimes v \otimes s}$, see e.g. [19]. The above proposition shows that $\text{Div } Q = \eta_0$. This result was proved in the more general case of skew-algebroids in [14].

6.3. Dirac generating operators for Lie bialgebroids. Let $(A, [\cdot, \cdot], \rho)$ and $(A^*, [\cdot, \cdot]_*, \rho_*)$ be two Lie algebroids, with homological vector fields $Q \in \mathfrak{X}(A[1])$ and $Q_* \in \mathfrak{X}(A^*[1])$ respectively. The Lie algebroid brackets, extended by the Leibniz rule, turn $(\mathcal{O}(A^*[1]), [\cdot, \cdot])$ and $(\mathcal{O}(A[1]), [\cdot, \cdot]_*)$ into Gerstenhaber algebras (see [18, 38]). The pair (A, A^*) is a Lie bialgebroid if Q_* is a derivation of the Gerstenhaber algebra $(\mathcal{O}(A^*[1]), [\cdot, \cdot])$, or equivalently if Q is a derivation of the Gerstenhaber algebra $(\mathcal{O}(A[1]), [\cdot, \cdot]_*)$.

The duality pairing between A and A^* extends to their exterior powers and leads to an isomorphism (see [8])

$$\beta_k : \wedge^k A \otimes (\wedge^n A^* \otimes \wedge^{\text{top}} T^* M)^{1/2} \longrightarrow \wedge^{n-k} A^* \otimes \mathcal{L}^{1/2},$$

where n is the rank of A and $0 \leq k \leq n$. In turn, the maps (β_k) induce an isomorphism

$$\beta : \Gamma(\text{Ber}_{A^*}^{1/2}) \xrightarrow{\sim} \Gamma(\text{Ber}_A^{1/2}).$$

By symmetry in A and A^* , the maps Φ_A , Υ_A and \mathcal{WQ}^A introduced in Section 6.1 have counterparts Φ_{A^*} , Υ_{A^*} and \mathcal{WQ}^{A^*} . We also denote by L_X the Lie derivative on $\Gamma(\text{Ber}_{A^*}^{1/2})$, along a vector field $X \in \mathfrak{X}(A^*[1])$.

As a consequence of Theorem 5.6, we recover the following main theorem of [8].

Theorem 6.9. [8] *Let (A, A^*) be a pair of Lie algebroids with homological vector fields $Q \in \mathfrak{X}(A[1])$ and $Q_* \in \mathfrak{X}(A^*[1])$, respectively. Let $F_Q \in \mathcal{O}(T^*[2](A[1]))$ and $F_{Q_*} \in \mathcal{O}(T^*[2](A^*[1]))$ be their corresponding Hamiltonian functions. Set $\Theta = F_Q + (\Phi_{A^*} \circ \Phi_A^{-1})^* F_{Q_*}$. The three following statements are equivalent:*

- (A, A^*) is a Lie bialgebroid,
- $\{\Theta, \Theta\} = 0$,
- $(L_Q + \beta \circ L_{Q_*} \circ \beta^{-1})^2 \in C^\infty(M)$.

Proof. The equivalence between the first two points is proved in [29].

We prove the equivalence between the last two points. By Theorem 5.6, $\mathcal{WQ}((\Phi_A)^* \Theta)$ is a Dirac generating operator if and only if $\{(\Phi_A)^* \Theta, (\Phi_A)^* \Theta\} = 0$, as a function on $T^*[2]M \oplus E[1]$. Since Υ_A is an algebra isomorphism and Φ_A is a symplectic diffeomorphism, we deduce that $\{\Theta, \Theta\} = 0$ if and only if $\mathcal{WQ}^A(\Theta)^2 \in C^\infty(M)$. The conclusion follows from the Lemma below. \square

Lemma 6.10. *Under the hypothesis of Theorem 6.9, we have $\mathcal{WQ}^A(\Theta) = L_Q + \beta \circ L_{Q_*} \circ \beta^{-1}$.*

Proof. By Proposition 6.4, we have $\mathcal{WQ}^A(F_Q) = L_Q$. By symmetry in A and A^* we also have $\mathcal{WQ}^{A^*}(F_{Q_*}) = L_{Q_*}$. Since $\Upsilon_A = \beta \circ \Upsilon_{A^*} \circ \beta^{-1}$, the conclusion follows. \square

According to Proposition 6.7, the above Dirac generating operator is also equal to

$$\mathcal{WQ}^A(\Theta) = \tilde{Q} + \beta \circ \tilde{Q}_* \circ \beta^{-1},$$

with \tilde{Q} being the differential on $\Gamma(\text{Ber}_A^{1/2})$ and \tilde{Q}_* the differential on $\Gamma(\text{Ber}_{A^*}^{1/2})$. This expression is the one obtained in [8].

Choosing $s \in \Gamma(\wedge^{\text{top}} T^*M)$, $v \in \Gamma(\wedge^{\text{top}} A)$ and $w \in \Gamma(\wedge^{\text{top}} A^*)$, we obtain modular cocycles $\eta_0 \in \Gamma(A^*)$ and $\zeta_0 \in \Gamma(A)$ of the Lie algebroids A and A^* respectively. We work locally and assume that $\langle v, w \rangle = 1$. Then, v provides a local isomorphism $v^\sharp : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^{n-k} A)$, and we define $\hat{Q}_* := (-1)^k (v^\sharp)^{-1} \circ Q_* \circ v^\sharp$ on $\Gamma(\wedge^k A^*)$, for all $k \in \mathbb{N}$. According to [8], we have

$$\mathcal{WQ}^A(\Theta) = \left(Q - \hat{Q}_* + \frac{1}{2}\eta_0 + \frac{1}{2}\gamma(\zeta_0) \right) \otimes \text{id}_{\mathcal{L}^{1/2}},$$

under the local trivialization of $\mathcal{L}^{1/2}$ provided by the section $(v \otimes s)^{1/2}$. As a consequence of Proposition 5.13, we obtain

Corollary 6.11. *The function $2\mathcal{WQ}^A(\Theta)^2 = \frac{1}{2}\langle \zeta_0, \eta_0 \rangle - \hat{Q}_*\eta_0$ is an invariant of the Lie bialgebroid (A, A^*) .*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JENA, GERMANY
 E-mail address: melchiorG@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LIÈGE, BELGIUM
 E-mail address: jean-philippe.michel@ulg.ac.be

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNITED STATES
 E-mail address: ping@math.psu.edu